



# Packings and Coverings of the Complete Graph with Trees

by

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# Abstract

In this thesis, we define the spectrum problem for packings (coverings) of  $G$  to be the problem of finding all graphs  $H$  such that a maximum  $G$ -packing (minimum  $G$ -covering) of the complete graph with the leave (excess) graph  $H$  exists. The set of achievable leave (excess) graphs in  $G$ -packings ( $G$ -coverings) of the complete graph is called the spectrum of leave (excess) graphs for  $G$ . Then, we consider this problem for trees with up to five edges.

We will prove that for any tree  $T$  with up to five edges, if the leave graph in a maximum  $T$ -packing of the complete graph  $K_n$  has  $i$  edges, then the spectrum of leave graphs for  $T$  is the set of all simple graphs with  $i$  edges. In fact, for these  $T$  and  $i$  and  $H$  any simple graph with  $i$  edges, we will construct a maximum  $T$ -packing of  $K_n$  with the leave graph  $H$ .

We will also show that for any tree  $T$  with  $k \leq 5$  edges, if the excess graph in a minimum  $T$ -covering of the complete graph  $K_n$  has  $i$  edges, then the spectrum of excess graphs for  $T$  is the set of all simple graphs and multigraphs with  $i$  edges, except for the case that  $T$  is a 5-star, for which the graph formed by four multiple edges is not achievable when  $n = 12$ .

To my parents

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# Statement of contribution

All the original work in this thesis was done in collaboration with my supervisor Dr. Dyer and my co-supervisor Dr. Shalaby.

In this thesis, we solve the spectrum problem for packings and coverings of the complete graph with trees that have up to five edges. We prove that all possible leave and excess graphs in packings and coverings of the complete graph with trees that have up to five edges are achievable, except for the four multiple edges which is not achievable in covering the complete graph on 12 vertices with 5-stars. Also we use new techniques in our proofs.

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# Chapter 1

## Introduction

### 1.1 History

Graph decompositions were first introduced by Plucker in 1835 [33]. He considered triangle-decompositions of the complete graph. In 1839, he realized that the necessary conditions for the existence of a triangle-decomposition of a complete graph on  $n$  vertices are  $n \equiv 1$  or  $3 \pmod{6}$  [34]. The generalization of this problem was stated by Woolhouse in 1844 as follows [50].

*Determine the number of combinations that can be made out of  $n$  symbols,  $p$  symbols in each; with this limitation, that no combination of  $q$  symbols which may appear in any of them may appear in any other.*

This problem is asking for the maximum size of a Steiner system with parameters  $q, p$ , and  $n$ . In 1847, Kirkman solved the problem for the case  $p = 3$  and  $q = 2$  [29]. In fact, he proved that the condition  $n \equiv 1$  or  $3 \pmod{6}$  is also sufficient for the existence of a triangle-decomposition of the complete graph on  $n$  vertices. Three years later, Kirkman posed and solved his schoolgirl problem, in which he considers resolvable triangle-decompositions of the complete graph [30].

*Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast.*

Decomposition of the complete graph with non-clique graphs was first mentioned by Dudeney in 1917 [12].

*Nine schoolboys walk out in triplets on the six week days so that no boy ever walks side by side with any other boy more than once. How would you arrange them?*

This problem involves resolvable path-decompositions of the complete graph.

The *spectrum problem for decomposition for a graph  $G$*  is to find necessary and sufficient conditions for  $n$  such that the complete graph on  $n$  vertices has a  $G$ -decomposition.

For any graph  $G$ , if a  $G$ -decomposition of the complete graph on  $n$  vertices exists, it is obviously necessary that the number of vertices of  $G$  be at most equal to  $n$ , the number of edges of the complete graph be a multiple of the number of edges of  $G$ , and the greatest common divisor of the vertex degrees of  $G$  divide the degree of each vertex of the complete graph, which is  $n - 1$ . In 1975, Wilson proved that for any graph  $G$ , there exists an integer  $N$  such that for any  $n \geq N$  which satisfies these necessary conditions, there is a  $G$ -decomposition of the complete graph on  $n$  vertices [49]. However, in order to solve the spectrum problem for decomposition for a graph  $G$  completely, it still remains to determine the specific conditions for  $n$  such that a  $G$ -decomposition of the complete graph on  $n$  vertices exists. Alternatively, what is the smallest value for  $N$  that makes the statement true?

Since 1835, the spectrum problem for decomposition has been considered for many graphs [1]. As in this thesis we are studying packings and coverings of the complete graph with trees, here we will concentrate on trees.

In 1964, Ringel published his famous conjecture about the spectrum problem for decomposition for trees, stating that for any tree  $T$  on  $k + 1$  vertices, the graph  $K_{2k+1}$  has a  $T$ -decomposition (see [36]).

In 1967 Rosa introduced graph labelings in order to solve this conjecture. These labelings are powerful tools not only for dealing with trees, but also for solving the spectrum problem for decomposition for other types of graphs.

Two of the most important graph labelings are  $\rho$ -labelings and  $\rho^+$ -labelings (see [5] for all graph labeling definitions). Many results relating to graph labelings in general

(and  $\rho$ -labelings in particular) can be found in Gallian's dynamic survey of labelings [19]. There are two main results on the existence of a  $G$ -decomposition of  $K_n$  the first of which was given by Rosa and the second by El-Zanati et. al. (see [43] and [16]).

**Theorem 1.1.1 [43]** *If a graph  $G$  has a  $\rho$ -labeling, then  $K_{2|E(G)|+1}$  has a  $G$ -decomposition.*

**Theorem 1.1.2 [16]** *If a graph  $G$  has a  $\rho^+$ -labeling, then  $K_n$  has a  $G$ -decomposition for all  $n \equiv 1 \pmod{2|E(G)|}$ .*

Several results for special types of  $G$  such as trees have been proved using these two theorems.

The maximum vertex distance of all pairs of vertices of a graph is called the *diameter* of the graph. Removing all the vertices of degree 1 of a graph gives a new graph called the *base* of the graph. A graph is called a *caterpillar* if its base is a path and a *lobster* is a graph whose base is a caterpillar. A *comet* is a graph obtained from a star by replacing each edge with a path of length  $k$  for some fixed  $k$ . A tree is *symmetric* if it can be rooted so that any two vertices in the same level have the same degree.

The following two theorems summarize the main results on the existence of tree-decompositions of  $K_n$ . One arises from  $\rho$ -labeling and the other arises from  $\rho^+$ -labelings.

**Theorem 1.1.3 [1]** *Let  $T$  be a tree belonging to one of the following families.*

- *Trees with at most 55 vertices (L. Brankovic and A. Rosa, private communication).*
- *Trees with at most 4 leaves [24].*
- *Trees of diameter at most 5 [23], [52].*
- *Symmetric trees [35].*

*Then  $K_{2|E(T)|+1}$  has a  $T$ -decomposition.*

**Theorem 1.1.4 [1]** *Let  $T$  be a tree belonging to one of the following families.*

- *Trees with at most 21 vertices [20].*
- *Trees of diameter at most 5 [16].*
- *Symmetric trees of diameter 4 [17].*
- *Caterpillars [43].*
- *Comets [16].*

*Then  $K_n$  has a  $T$ -decomposition for all  $n \equiv 1 \pmod{2|E(T)|}$ .*

The following theorems are a few other results in the approach to the solution of the spectrum problem for decomposition for some classes of trees.

**Theorem 1.1.5 [25]** *Let  $T$  be a caterpillar or lobster with  $m + 1$  vertices. If  $n \equiv 0$  or  $1 \pmod{2m}$ , then  $K_n$  has a  $T$ -decomposition. Moreover, if  $m = 2^\alpha$  for some integer  $\alpha \geq 0$ , then  $n \equiv 0$  or  $1 \pmod{2m}$  is also necessary for existence.*

**Theorem 1.1.6 [25]** *Let  $T$  be a tree with  $m + 1$  vertices. If  $T$  contains a vertex of degree  $d$  such that  $d \geq \frac{1}{2}(m + 3)$ , then  $K_{m+1}$  does not have a  $T$ -decomposition.*

**Theorem 1.1.7 [11]** *Let  $T$  be a tree with  $n + 1$  vertices, let  $x$  be a vertex in  $T$  and suppose either of the following holds.*

- *The graph obtained from  $T$  by removing  $x$  (and all the edges incident with  $x$ ) has at least  $n - \frac{\sqrt{n}}{4+2\sqrt{2}}$  isolated vertices.*
- *For a non-negative integer  $d$ , the diameter of  $T$  is at most  $d + 2$  and the graph obtained from  $T$  by removing  $x$  has at least  $n - cn$  isolated vertices where  $c = (\sqrt{1 + (4 + 4d)^2} - 4 - 4d)^2$ .*

*Then  $K_{2n+1}$  has a  $T$ -decomposition.*

Paths and stars are two infinite classes of trees for which the spectrum problem for decomposition is completely solved [46, 51].

**Theorem 1.1.8 [47]** *If  $n$  and  $m \geq 2$  are positive integers, then  $K_n$  has a  $P_m$ -decomposition if and only if  $n = 1$  or  $n \geq m$ , and  $n(n - 1) \equiv 0 \pmod{2m - 2}$ .*

**Theorem 1.1.9 [51]** *If  $n$  and  $k \geq 1$  are positive integers, then  $K_n$  has an  $S_k$ -decomposition if and only if  $n = 1$  or  $n \geq 2k$ , and  $n(n-1) \equiv 0 \pmod{2k}$ .*

The decomposition result which will be used the most in this thesis is the one established by Huang and Rosa in 1978 (Theorem 1.3.1), where they solved the spectrum problem for decomposition for trees with up to eight edges [25]. In fact, they proved that for any tree  $T$  with up to eight edges, a complete graph has a  $T$ -decomposition if and only if the number of edges of the complete graph is a multiple of the number of edges of  $T$ .

Up to this point, we have discussed graph decompositions. But what can we say when no decomposition exists? In this case, getting as close as possible to a decomposition is still desirable. This leads to packing and covering problems. The earliest result on packing was established by Kirkman in 1847, when he considered the problem of packing the complete graph with triangles [29]. In fact, he found the number of triangles in a maximum triangle-packing of the complete graph on any number of vertices.

Since 1847, the packing and covering problems have been considered for many graphs, the early results of which were collected in a survey paper by Beineke in 1969 [3].

In 1997 and 1998, Caro and Yuster established a Wilson-like result for the packing and covering problems. In fact, they proved that for any graph  $H$  with  $h$  edges, there exists a positive integer  $n_0(H)$  such that for all integers  $n > n_0(H)$  the  $H$ -packing number of  $K_n$  is  $\lfloor \frac{dn}{2h} \lfloor \frac{n-1}{d} \rfloor \rfloor$ , where  $d$  is the greatest common divisor of all degrees of  $H$ , unless  $n \equiv 1 \pmod{d}$  and  $\frac{n(n-1)}{d} \equiv b \pmod{\frac{2h}{d}}$  where  $1 \leq b \leq d$  in which case the packing number is  $\lfloor \frac{dn}{2h} \lfloor \frac{n-1}{d} \rfloor \rfloor - 1$  [9]. They also proved that for any graph  $H$  with  $h$  edges, there exists a positive integer  $n_0(H)$  such that for all integers  $n > n_0(H)$  the  $H$ -covering number of  $K_n$  is  $\lceil \frac{dn}{2h} \lceil \frac{n-1}{d} \rceil \rceil$ , where  $d$  is the greatest common divisor of all degrees of  $H$ , unless  $d$  is even,  $n \equiv 1 \pmod{d}$  and  $\frac{n(n-1)}{d} + 1 \equiv 0 \pmod{\frac{2h}{d}}$ , in which case the covering number is  $\left\lceil \frac{\binom{n}{2}}{h} \right\rceil + 1$  [10].

However, in order to solve the packing and covering problems completely, it is required to find the smallest possible number  $n_0(H)$  in Caro and Yuster's results. Since 1969, the packing and covering problems have been solved for many other graphs. For instance, in 1999, Bryant et. al. considered packing and covering the complete graph



with cubes [2].

In papers published from 1975 to 1994, the problem of packing the complete graph with cycles of the same length was settled for cycles of length at most six [45, 44, 26, 27]. Moreover, for cycles of lengths four and six, the covering problem is solved as well [45, 28]. In 2008, Bryant and Horsley considered a generalization of the packing problem for cycles and solved the problem of packing the complete graph with cycles of arbitrary specified length [6].

The packing problem has also been considered for paths. In 1983, Tarsi conjectured that the necessary and sufficient conditions for the existence of a packing of the complete multigraph on  $n$  vertices with multiplicity  $\lambda$  with paths of arbitrary specified lengths, are that the length of each path is at most  $n - 1$  and the sum of the lengths is at most the number of edges of the complete multigraph [47]. He proved his conjecture for odd  $n$ , even  $\lambda$ , and each length being at most  $n - 3$ . In 2009, Bryant proved Tarsi's conjecture for the general case [4].

The problem of packing and covering the complete graph with trees that have up to six edges was solved by Roditty [37, 38, 39, 40]. In fact, for any tree  $T$  with up to six edges, Roditty found a maximum  $T$ -packing of the complete graph on any number of vertices with a leave graph whose edges could be covered by adding one more tree  $T$  to the packing. Using this method, he could obtain maximum  $T$ -packings and minimum  $T$ -coverings simultaneously. He proved that except for some small  $n$ , the number of trees  $T$  in a  $T$ -packing ( $T$ -covering) of the complete graph on  $n$  vertices is equal to  $\left\lfloor \frac{n(n-1)}{2i} \right\rfloor \left( \left\lceil \frac{n(n-1)}{2i} \right\rceil \right)$ , where  $T$  is any tree with  $i$  edges,  $i \leq 6$ .

In papers published from 1993 to 1997, Kennedy solved the problem of packing and covering the complete graph with hexagons. Moreover, she found necessary and sufficient conditions for the existence of every possible leave and excess graph [26, 27, 28].

In this thesis, we introduce the spectrum problem for packing (covering), which is to determine the set of all achievable leave (excess) graphs in maximum packings (minimum coverings) of the complete graph with isomorphic graphs  $G$ . As stated above, Kennedy in fact solved the spectrum problem for packing and covering for hexagons. We consider these problems for trees with up to five edges, and prove that all possible leave (excess) graphs in maximum packings (minimum coverings) of the complete graph with trees that have up to five edges, are achievable. However,

the graph formed by four multiple edges cannot be obtained as the excess graph in covering the complete graph on 12 vertices with 5-stars.

In the first chapter, we see the history behind this work as well as definitions and preliminaries which will be needed in the next chapters.

The second chapter contains two sections. The first section consists of two main theorems which state that all possible leave and excess graphs in maximum packings and minimum coverings of the complete graph with 4-stars are achievable. The results of this section are accepted for publication in the Journal of Combinatorial Mathematics and Combinatorial Computation [13]. The second section also consists of two theorems which state the same results for 5-stars, except for the graph formed by four multiple edges which is not achievable as the excess graph in covering the complete graph on 12 vertices with 5-stars. The results of this section are published in Graphs and Combinatorics [15].

The third chapter contains two main theorems which state that all leave and excess graphs in maximum packings and minimum coverings of the complete graph with trees that have up to five edges are achievable. The results of this chapter are accepted for publication in the Journal of Combinatorial Mathematics and combinatorial computation [14].

Finally, in the fourth chapter we will summarize the results and discuss future directions.

## 1.2 Basic Definitions

A *graph*  $G = (V, E)$  is formed by a finite set  $V$  of *vertices* and a set  $E$  of *edges* joining pairs of distinct vertices. The vertices  $u$  and  $v$  are called *adjacent* if there is an edge between them and the edge is denoted by  $\{u, v\}$ . If there is at most one edge between every pair of vertices, the graph is called a *simple graph*. In this thesis, we assume the graphs are simple, unless otherwise stated. If there is more than one edge between two vertices, the graph is called a *multigraph* and those edges are called *multiple edges*. The *degree* of a vertex is the number of edges which are incident with that vertex. A vertex is *isolated* if its degree is zero.

A *complete graph* on  $n$  vertices, denoted  $K_n$ , is a graph on  $n$  vertices where all

pairs of vertices are adjacent.

We call a graph  $G = (V, E)$  *bipartite* if  $V$  admits a partition into two classes such that every edge has its ends in different classes. A bipartite graph in which every two vertices from different partition classes are adjacent is a *complete bipartite graph*. A complete bipartite graph with class sizes  $m$  and  $n$  is denoted by  $K_{m,n}$ .

For each positive integer  $n$  the complete bipartite graph  $S_n = K_{1,n}$  is called an *n-star*. The vertex of degree  $n$  is the *center* and the vertices of degree 1 are the *leaves* of the star.

A *path*  $P$  is a sequence of distinct vertices with each pair of consecutive vertices in  $P$  joined by an edge. We denote a path on  $n$  vertices by  $P_n$ . If we join the first and last vertex on this path, we call it a *cycle*. A cycle on  $n$  vertices is denoted by  $C_n$ . A *connected graph* is a graph with at least one path between each pair of vertices. A *tree* is a connected graph which contains no cycles.

Two graphs  $G$  and  $G'$  are called *isomorphic* if there exists a one-to-one correspondence between the vertices in  $G$  and the vertices in  $G'$  such that a pair of vertices are adjacent in  $G$  if and only if the corresponding pair of vertices are adjacent in  $G'$ . Such a one-to-one correspondence of vertices that preserves adjacency is called an *isomorphism*. For example, the graphs in Figure 1.1 are two isomorphic 5-cycles.

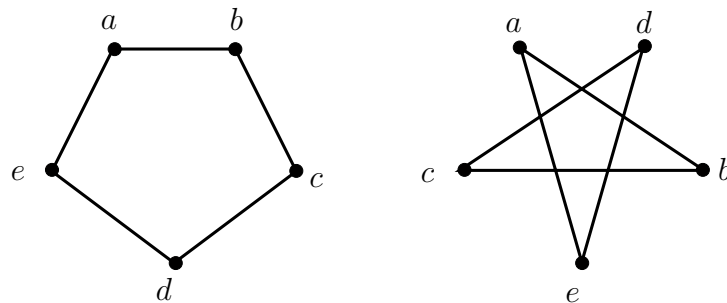


Figure 1.1: Two isomorphic graphs

Let  $m$  and  $n$  be positive integers. The *disjoint union* of graphs  $G$  and  $H$ , denoted  $G + H$ , is a graph with the union of vertex sets of  $G$  and  $H$  as its vertex set and the union of the edge sets of  $G$  and  $H$  as its edge set. The *join* of simple graphs  $G$  and  $H$ , denoted  $G \vee H$  is the graph obtained from the disjoint union  $G + H$  by adding the edges  $\{\{x, y\} | x \in V(G), y \in V(H)\}$ . Also for any graph  $G$ ,  $mG$  is the graph

consisting of  $m$  pairwise disjoint copies of  $G$ . Furthermore, we denote the complete multigraph on  $n$  vertices with multiplicity  $m$  by  $K_n^m$  following West [48].

Consider the graph  $G \vee H$  where  $G$  and  $H$  are graphs on  $m$  and  $n$  vertices, respectively. A  $(\mathbb{Z}_m, \mathbb{Z}_n)$ -*labeling* of the vertices of the graph  $G \vee H$  is a labeling such that the vertices of  $G$  are labelled with the elements of  $\mathbb{Z}_m$  having subscript 1 and the vertices of  $H$  are labelled with the elements of  $\mathbb{Z}_n$  having subscript 2.

For graphs  $G$  and  $H$ , a  $G$ -*decomposition* of  $H$  is a partition of the edge set of  $H$  into graphs isomorphic to  $G$ . A  $G$ -*design* of order  $n$  is a  $G$ -decomposition of the complete graph  $K_n$ . The *spectrum problem for decomposition* for a graph  $G$  is to find necessary and sufficient conditions on  $n$  such that a  $G$ -design of order  $n$  exists, and the *spectrum of decomposition for a graph  $G$*  is the set of integers satisfying those conditions.

For graphs  $G$  and  $H$ , a  $G$ -*packing* of  $H$  is a set of subgraphs of  $H$  all isomorphic to  $G$ , such that each edge of  $H$  is contained in at most one subgraph. Let  $\mathbb{P}$  be a  $G$ -packing of  $H$  and  $P$  be the graph with vertex set  $V(H)$  and edge set the union of the edges of all subgraphs in  $\mathbb{P}$ . The non-isolated vertices of the graph  $H \setminus P$  together with the edge set of this graph forms a graph called the *leave graph*. Hence, if  $H$  is a simple graph, then the leave graph is also a simple graph. A *maximum  $G$ -packing* of  $H$  is a  $G$ -packing with the smallest possible number of edges in the leave graph.

A  $G$ -*covering* of  $H$  is a set of subgraphs  $G$  of  $H$  whose union is  $H$ . Let  $\mathbb{C}$  be a  $G$ -covering of  $H$  and  $C$  be the graph with vertex set  $V(H)$  and edge set the union of the edges of all subgraphs in  $\mathbb{C}$ . Consider the set of edges of  $C$  a multiset with the multiplicity of each edge  $e$  being the number of subgraphs that include  $e$ . Then the graph  $C \setminus H$  is called the *excess graph*. Hence, the excess graph is possibly a multigraph even when  $H$  is simple. A *minimum covering* of  $K_n$  with isomorphic graphs  $G$  is a covering with the smallest number of edges in the excess graph. Note that in a packing, every edge exists in at most one subgraph, while in the covering every edge exists at least in one subgraph.

Figures 1.2 and 1.3 illustrate a  $K_3$ -packing and a maximum  $K_3$ -packing of  $K_6$ , with the leave graphs  $K_{3,3}$  and  $3K_2$ , respectively.

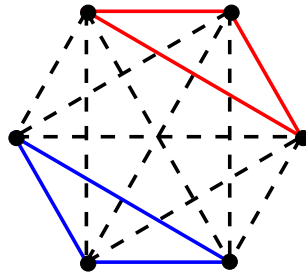
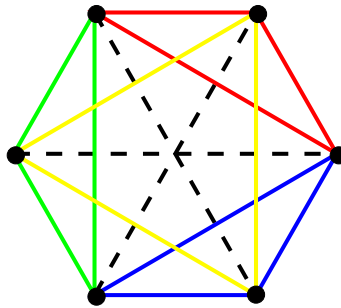
Figure 1.2: A  $K_3$ -packing of  $K_6$ Figure 1.3: A maximum  $K_3$ -packing of  $K_6$ 

Figure 1.4 demonstrates a  $K_3$ -covering of  $K_6$  with the edges  $\{1, 2\}$  (twice),  $\{0, 2\}$ ,  $\{1, 5\}$ ,  $\{2, 3\}$ , and  $\{2, 4\}$  as the edges of the excess graph.

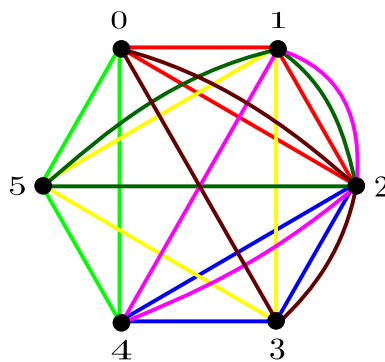
Figure 1.4: A  $K_3$ -covering of  $K_6$ 

Figure 1.5 represents a minimum  $K_3$ -covering of  $K_6$ , with the edges  $\{0, 1\}$ ,  $\{2, 3\}$ , and  $\{4, 5\}$  as the edges of the excess graph.

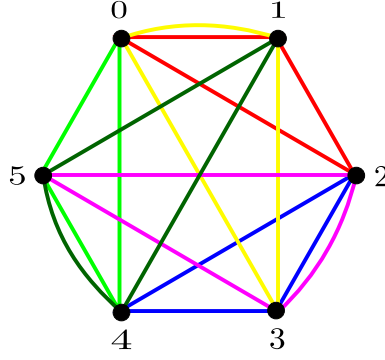


Figure 1.5: A minimum  $K_3$ -covering of  $K_6$

The  $G$ -packing number ( $G$ -covering number) of  $H$  is the number of graphs  $G$  in a maximum  $G$ -packing (minimum  $G$ -covering) of  $H$ . The  $G$ -packing ( $G$ -covering) problem of a graph  $H$  is to determine the  $G$ -packing number ( $G$ -covering number) of  $H$ . Roditty solved the problem for all trees with up to six edges [37, 38, 39, 40].

Different packings (coverings) might lead to different leave (excess) graphs, even in the case of maximum packings (minimum coverings). The *spectrum problem for packing (covering) for a graph  $G$*  is to determine the set of all achievable leave (excess) graphs in maximum  $G$ -packings (minimum  $G$ -coverings) of the complete graph. We call this set the *spectrum of leave (excess) graphs for  $G$* . We consider these problems for trees with up to five edges. In fact, we prove if the leave graph in a maximum  $T$ -packing of any complete graph has  $i$  edges, then the spectrum of leave graphs for  $T$  is the set of all simple graphs with  $i$  edges, when  $T$  is any tree with up to five edges. We also prove that for any tree  $T$  with up to five edges, if the excess graph in a minimum  $T$ -covering of the complete graph has  $i$  edges, then the spectrum of excess graphs for  $T$  is the set of all simple graphs and multigraphs with  $i$  edges, except for graph  $K_2^4$  which cannot be the excess graph in any  $S_5$ -covering of  $K_{12}$ .

### 1.3 Preliminary Results

In this section, we will present several lemmas which will be used in the proofs of the theorems in the next chapters.

The non-isomorphic trees with three edges are  $S_3$  and  $P_4$ , the non-isomorphic trees with four edges are  $S_4$ ,  $P_5$ , and  $A$ , and the non-isomorphic trees with five edges are

$S_5, B, C, D, E$ , and  $P_6$  as shown in Figure 1.6.

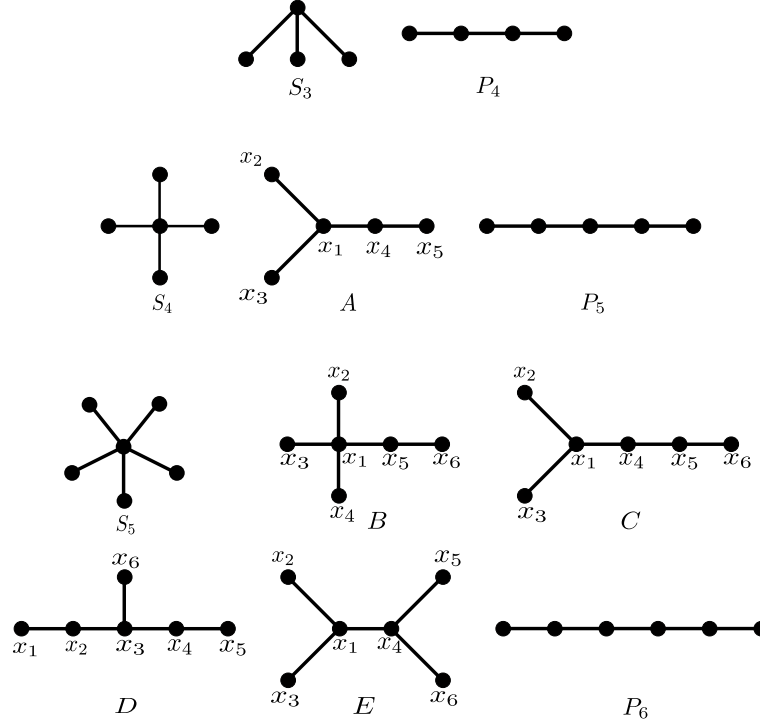


Figure 1.6: All non-isomorphic trees with three, four, or five edges

**Notation.** For any integer  $k$ , we denote the star  $S_k$  with the center  $x$  and leaves  $y_1, y_2, \dots, y_k$  by  $(x; y_1, y_2, \dots, y_k)$ . Also for any integer  $k$ , we denote the path  $P_k$  with the sequence of vertices  $x_1, x_2, \dots, x_k$  by  $(x_1, x_2, \dots, x_k)$ . Now consider the vertex labels in Figure 1.6. We denote the trees  $A, B, C, D$ , and  $E$  with  $(x_1; x_2, x_3, x_4 - x_5)$ ,  $(x_1; x_2, x_3, x_4, x_5 - x_6)$ ,  $(x_1; x_2, x_3, x_4 - x_5 - x_6)$ ,  $(x_3; x_6, x_2, x_4 - x_1, x_5)$ , and  $(x_1 - x_2, x_3; x_4 - x_5, x_6)$ , respectively.

In 1978 Huang and Rosa [25] solved the spectrum problem for decomposition for trees with up to eight edges.

**Theorem 1.3.1 [25]** *If  $n$  is any positive integer and  $T$  is any tree with  $i$  edges, where  $i \leq 8$ , then the complete graph  $K_n$  has a  $T$ -decomposition if and only if  $\frac{n(n-1)}{2} \equiv 0 \pmod{i}$ .*

Roditty solved the packing and covering problems for trees with up to six edges [37, 38, 39, 40].

**Theorem 1.3.2** [37, 38, 39, 40] *If  $T$  is a tree with  $i$  edges where  $i \leq 6$  and  $n \geq 2i - 1$  is any integer, then the  $T$ -packing number of  $K_n$  is  $\left\lfloor \frac{n(n-1)}{2i} \right\rfloor$  and the number of edges in the leave graph of a maximum  $T$ -packing of  $K_n$  is  $\frac{n(n-1)}{2} - i \left\lfloor \frac{n(n-1)}{2i} \right\rfloor$ .*

**Theorem 1.3.3** [37, 38, 39, 40] *If  $T$  is a tree with  $i$  edges where  $i \leq 6$  and  $n \geq 2i$  is any integer, then the  $T$ -covering number of  $K_n$  is  $\left\lceil \frac{n(n-1)}{2i} \right\rceil$  and the number of edges in the excess graph of a minimum  $T$ -covering of  $K_n$  is  $i \left\lceil \frac{n(n-1)}{2i} \right\rceil - \frac{n(n-1)}{2}$ .*

We will use the following lemmas in the proof of our main theorems.

**Lemma 1.3.4** *Let  $s$  be a positive odd integer and  $sK_2$  be the union of  $s$  disjoint edges. For positive integers  $s$  and  $t$  with  $s \leq t$  the complete bipartite graph  $K_{t,s}$  can be packed with  $(t-1)$ -stars with an  $sK_2$  as the leave graph.*

**Proof.** Label the vertices of  $K_{t,s}$  with a  $(\mathbb{Z}_t, \mathbb{Z}_s)$ -labeling. The following stars form a maximum  $S_{t-1}$ -packing of  $K_{t,s}$  with the  $s$  edges  $\{0_1, 1_2\}, \{1_1, 2_2\}, \dots, \{(s-2)_1, (s-1)_2\}$ , and  $\{(t-1)_1, 0_2\}$  as the leave graph (see Figure 1.7). For numbers with subscript 1 the computations are done modulo  $t$  and for those with subscript 2 the computations are done modulo  $s$ .

$$(i_2; i_1, (i+1)_1, \dots, (i+t-2)_1), \quad i = 0, 1, \dots, s-1.$$

■

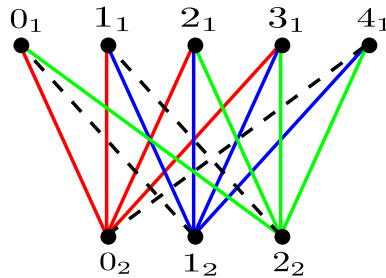


Figure 1.7: An  $S_4$ -packing of  $K_{3,5}$  with the leave graph  $3K_2$

The proof to Lemma 1.3.5 is trivial.



**Lemma 1.3.5** *If  $m$ ,  $n$ , and  $k$  are positive integers, then the complete bipartite graph  $K_{m,kn}$  has an  $S_k$ -decomposition.*

**Lemma 1.3.6** *If  $k$  is a positive integer and  $s$  is a positive odd integer, then the graph  $K_s \vee \frac{(k-1)(s-1)}{2} K_1$  has an  $S_k$ -decomposition.*

**Proof.** Let  $k$  be a positive integer and  $s$  be a positive odd integer. Label the vertices of the graph  $K_s \vee \frac{(s-1)(k-1)}{2} K_1$  with a  $(\mathbb{Z}_s, \mathbb{Z}_{\frac{(s-1)(k-1)}{2}})$ -labeling. Then, the following stars form an  $S_k$ -decomposition for  $K_s \vee \frac{(k-1)(s-1)}{2} K_1$  where  $i \in \mathbb{Z}_s$  and  $j = 0, 1, \dots, \frac{s-3}{2}$  (see Figure 1.8).

$$(i_1; (i+j+1)_1, ((k-1)j)_2, ((k-1)j+1)_2, ((k-1)j+2)_2, \dots, ((k-1)j+k-2)_2)$$

■

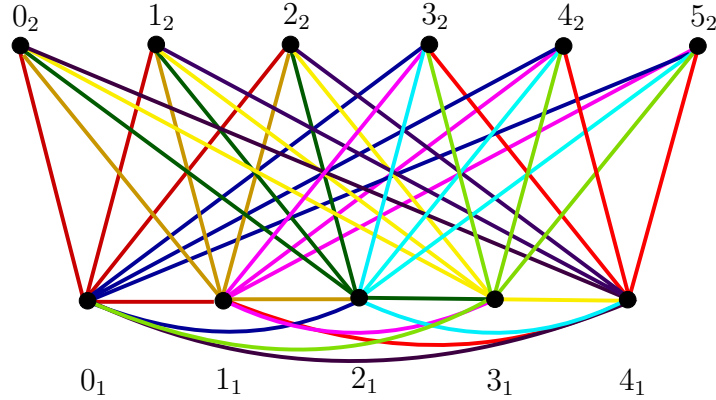


Figure 1.8: An  $S_4$ -decomposition of  $K_5 \vee 6K_1$

**Lemma 1.3.7** *If  $k$  is a positive integer,  $s$  is a positive odd integer, and  $k \geq \frac{s-1}{2}$ , then the graph  $K_s \vee \frac{2k-s+1}{2} K_1$  has an  $S_k$ -decomposition.*

**Proof.** Let  $k$  be a positive integer and  $s$  be a positive odd integer such that  $k \geq \frac{s-1}{2}$ . Label the vertices of  $K_s \vee \frac{2k-s+1}{2} K_1$  with a  $(\mathbb{Z}_s, \mathbb{Z}_{\frac{2k-s+1}{2}})$ -labeling. The following stars will form an  $S_k$ -decomposition for the graph  $K_s \vee \frac{2k-s+1}{2} K_1$  (see Figure 1.9).

$$\left( i_1; (i+1)_1, (i+2)_1, \dots, \left( i + \left( \frac{s-1}{2} \right) \right)_1, 0_2, 1_2, \dots, \left( \frac{2k-s-1}{2} \right)_2 \right), i \in \mathbb{Z}_s.$$

■

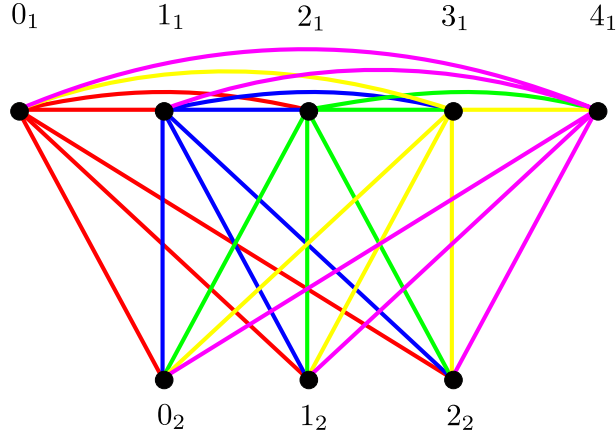


Figure 1.9: An  $S_5$ -decomposition of  $K_5 \vee 3K_1$

**Corollary 1.3.8** *If  $k \geq 2$  is a positive integer, then the graph  $K_{2k-1}$  has a maximum  $S_k$ -packing with a single edge as the leave graph.*

**Proof.** For  $k = 2$ ,  $K_{2k-1}$  is a triangle and the result follows immediately. For  $k > 2$ , write  $K_{2k-1} = K_{2k-3} \vee K_2$ . Letting  $s = 2k - 3$ , the result follows by Lemma 1.3.7. ■

**Corollary 1.3.9** *If  $n$  and  $k$  are positive integers such that  $n \equiv 2k - 1 \pmod{2k}$ , then  $K_n$  has a maximum packing with  $k$ -stars with a single edge as the leave graph.*

**Proof.** Let  $n$  be a positive integer such that  $n \equiv 2k - 1 \pmod{2k}$ . If  $n = 2k - 1$ , the result follows from Corollary 1.3.8. If  $n > 2k - 1$ , then write  $K_n = K_{2k-1} \vee K_{n-2k+1}$ . Since  $n \equiv 2k - 1 \pmod{2k}$ ,  $K_{n-2k+1}$  has an  $S_k$ -decomposition,  $R$ , by Theorem 1.1.9. Moreover,  $K_{2k-1}$  has a maximum packing  $S$  with  $k$ -stars with a single edge as the leave graph, by Corollary 1.3.8. Now,  $K_{2k-1, n-2k+1}$  is a complete bipartite graph with one part of size a multiple of  $k$  and hence, it has an  $S_k$ -decomposition  $T$  by Lemma 1.3.5. Therefore,  $R \cup S \cup T$  is a maximum  $S_k$ -packing of  $K_n$  with a single edge as the leave graph. ■

The following lemmas will greatly reduce the number of cases in the proofs of our main theorems.

**Lemma 1.3.10** *If  $k$  is a positive odd integer,  $n \geq \frac{k+1}{2}$  is an integer, and  $H$  is the leave graph (excess graph) in an  $S_k$ -packing ( $S_k$ -covering) of the complete graph  $K_n$ , then*

there exists an  $S_k$ -packing ( $S_k$ -covering) of  $K_{n+k}$  with  $H$  as the leave graph (excess graph).

**Proof.** Let  $k$  be a positive odd integer and  $n \geq \frac{k+1}{2}$  be a positive integer. Write  $K_{n+k} = K_n \vee K_k$ . Let  $R$  be an  $S_k$ -packing of  $K_n$  with the leave graph  $H$ . Label the vertices of  $K_n \vee K_k$  with a  $(\mathbb{Z}_n, \mathbb{Z}_k)$ -labeling. The set of vertices  $\{0_1, 1_1, \dots, (\frac{2n-k-3}{2})_1\}$ , the set of vertices  $\{0_2, 1_2, \dots, (k-1)_2\}$ , and the edges between these two sets form a complete bipartite graph with one part of size a multiple of  $k$ . Hence, by Lemma 1.3.5, this complete bipartite graph has an  $S_k$ -decomposition,  $S$ . Moreover, the set of vertices  $(\frac{2n-k-1}{2})_1, (\frac{2n-k+1}{2})_1, \dots, (n-1)_1$ , the set of vertices  $0_2, 1_2, \dots, (k-1)_2$ , the edges between these two sets, and the edges between the vertices of the second set form a  $K_k \vee \frac{k+1}{2}K_1$ . Hence, by Lemma 1.3.7, the graph  $K_k \vee \frac{k+1}{2}K_1$  has an  $S_k$ -decomposition,  $T$ . Therefore,  $R \cup S \cup T$  forms an  $S_k$ -packing of  $K_{n+k}$  with  $H$  as the leave graph.

The proof is similar for the covering case. ■

**Lemma 1.3.11** *If  $k$  and  $n$  are positive integers such that  $n \geq 2k$ , and  $H$  is the leave graph (excess graph) in an  $S_k$ -packing ( $S_k$ -covering) of the complete graph  $K_n$ , then there exists an  $S_k$ -packing ( $S_k$ -covering) of  $K_{n+2k}$  with  $H$  as the leave graph (excess graph).*

**Proof.** Let  $k$  and  $n$  be positive integers such that  $n \geq 2k$ . Write  $K_{n+2k} = K_{2k} \vee K_n$ . Let  $R$  be an  $S_k$ -packing of  $K_n$  with the leave graph  $H$ . The graph  $K_{2k}$  has an  $S_k$ -decomposition,  $S$ , by Theorem 1.1.9. Moreover, the set of vertices of  $K_n$ , the set of vertices of  $K_{2k}$ , and the edges between these two sets form a complete bipartite graph with one part of size a multiple of  $k$ . Hence, this graph has an  $S_k$ -decomposition,  $T$ , by Lemma 1.3.5. Therefore,  $R \cup S \cup T$  forms an  $S_k$ -packing of  $K_{n+2k}$  with the leave graph  $H$ .

The proof is similar for the covering case. ■

Note that the Lemmas 1.3.10 and 1.3.11 work for maximum packings and minimum coverings as particular cases, but also work for decompositions.

**Lemma 1.3.12** *If  $m$  and  $n$  are positive integers and  $n \geq 2$ , then the graph  $K_{3m,n}$  has a  $P_4$ -decomposition.*

**Proof.** Let  $m$  and  $n \geq 2$  be positive integers. In order to prove the result, it suffices to show that  $K_{3,2}$  and  $K_{3,3}$  have  $P_4$ -decompositions.

For  $K_{3,2}$ , label the vertices with a  $(\mathbb{Z}_3, \mathbb{Z}_2)$ -labeling. The following paths form a  $P_4$ -decomposition of  $K_{3,2}$ .

$$(0_1, 0_2, 1_1, 1_2), (0_1, 1_2, 2_1, 0_2)$$

For  $K_{3,3}$ , label the vertices with a  $(\mathbb{Z}_3, \mathbb{Z}_3)$ -labeling. The following paths form a  $P_4$ -decomposition of  $K_{3,3}$ .

$$(0_1, 0_2, 1_1, 1_2), (0_2, 2_1, 2_2, 1_1), (2_1, 1_2, 0_1, 2_2)$$

■

**Lemma 1.3.13** *If  $n \geq 2$  is an integer and  $H$  is the leave (excess) graph in a  $P_4$ -packing ( $P_4$ -covering) of  $K_n$ , then there exists a  $P_4$ -packing ( $P_4$ -covering) of  $K_{n+3}$  with the leave (excess) graph  $H$ .*

**Proof.** Let  $n \geq 2$  be an integer and  $R$  be a maximum  $P_4$ -packing of  $K_n$  with the leave graph  $H$ . Write  $K_{n+3} = K_n \vee K_3$ . Label the vertices of  $K_n \vee K_3$  with a  $(\mathbb{Z}_n, \mathbb{Z}_3)$ -labeling. The set of vertices  $\{2_1, 3_1, \dots, (n-1)_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , and the edges between these two sets form a graph  $K_{n-2,3}$  which has a  $P_4$ -decomposition,  $S$ , by Lemma 1.3.12. The set of vertices  $\{0_1, 1_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges between these two sets, and the edges within the latter, form a graph  $2K_1 \vee K_3$ . The following paths form a  $P_4$ -decomposition,  $T$ , for this graph.

$$(0_1, 0_2, 1_2, 1_1), (0_1, 1_2, 2_2, 1_1), (0_1, 2_2, 0_2, 1_1)$$

Therefore,  $R \cup S \cup T$  forms a maximum  $P_4$ -packing of  $K_{n+3}$  with the leave graph  $H$ .

The proof of the covering case uses a similar argument. ■

**Lemma 1.3.14** *For positive integers  $m$  and  $n$ ,  $n \geq 2$ , the graph  $K_{4m,n}$  has a  $T$ -decomposition for any tree  $T$  with four edges.*

**Proof.** Let  $m$  and  $n \geq 2$  be positive integers and  $T$  be any tree with four edges. The result is immediate for the case that  $T$  is the star  $S_4$ , by Lemma 1.3.5.

Now consider  $T$  as the tree  $A$ . It suffices to show that  $K_{4,2}$  and  $K_{4,3}$  have  $A$ -decompositions. For  $K_{4,2}$ , label the vertices of  $K_{4,2}$  with a  $(\mathbb{Z}_4, \mathbb{Z}_2)$ -labeling. Then, the following trees form an  $A$ -decomposition of the graph  $K_{4,2}$ . (See Figure 1.10.)

$$(0_2; 0_1, 1_1, 2_1 - 1_2), (1_2; 0_1, 1_1, 3_1 - 0_2)$$

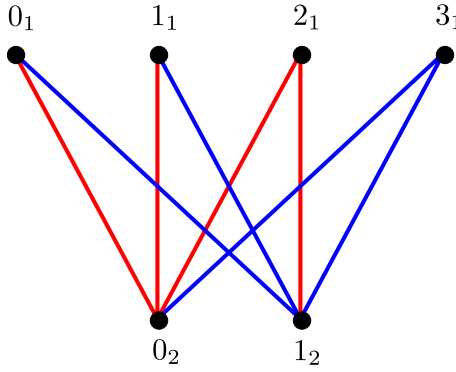


Figure 1.10: An  $A$ -decomposition of  $K_{4,2}$

For  $K_{4,3}$ , label the vertices of  $K_{4,3}$  with a  $(\mathbb{Z}_4, \mathbb{Z}_3)$ -labeling. Then, the following trees form an  $A$ -decomposition of the graph  $K_{4,3}$  (See Figure 1.11).

$$(0_1; 0_2, 1_2, 2_2 - 1_1), (2_1; 0_2, 2_2, 1_2 - 1_1), (3_1; 1_2, 2_2, 0_2 - 1_1)$$

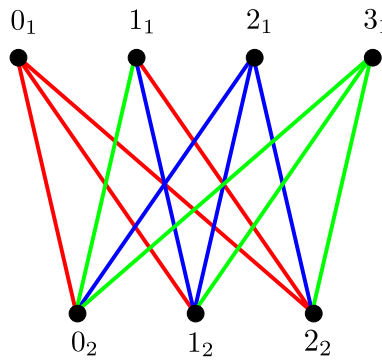


Figure 1.11: An  $A$ -decomposition of  $K_{4,3}$

Now consider  $T$  as the path  $P_5$ . For  $K_{4,2}$ , label the vertices with a  $(\mathbb{Z}_4, \mathbb{Z}_2)$ -labeling.

The following paths form a  $P_5$ -decomposition for the graph  $K_{4,2}$ .

$$(0_1, 0_2, 1_1, 1_2, 2_1), (0_1, 1_2, 3_1, 0_2, 2_1)$$

For  $K_{4,3}$ , label the vertices with a  $(\mathbb{Z}_4, \mathbb{Z}_3)$ -labeling. The following paths form a  $P_5$ -decomposition of  $K_{4,3}$ .

$$(0_1, 0_2, 1_1, 1_2, 2_1), (0_1, 2_2, 2_1, 0_2, 3_1), (0_1, 1_2, 3_1, 2_2, 1_1)$$

■

**Corollary 1.3.15** *If  $n \geq 1$  is an integer,  $T$  any tree with four edges, and  $K_n$  has a  $T$ -packing with the leave graph  $H$ , then  $K_{n+8}$  has a  $T$ -packing with the leave graph  $H$ .*

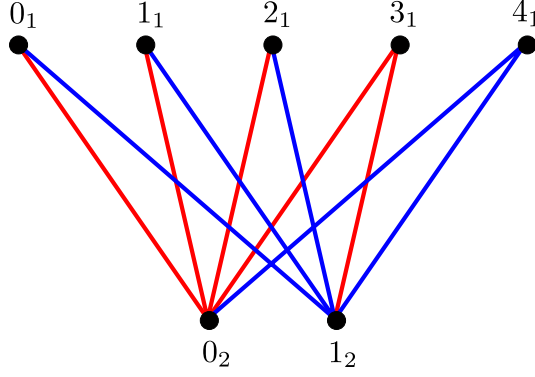
**Proof.** Let  $n \geq 1$ ,  $T$  be any tree with four edges, and  $R$  be a  $T$ -packing of  $K_n$  with the leave graph  $H$ . Write  $K_{n+8} = K_n \vee K_8$ . By Theorem 1.3.1, the complete graph  $K_8$  has a  $T$ -decomposition,  $S$ . Moreover, the complete bipartite graph  $K_{n,8}$  has a  $T$ -decomposition,  $U$ , by Lemma 1.3.14. Therefore,  $R \cup S \cup U$  forms a  $T$ -packing of  $K_{n+8}$  with the leave graph  $H$ . ■

Lemmas 1.3.16 and 1.3.17 will be used to prove that all possible leave graphs in  $T$ -packings of  $K_n$  are achievable, where  $T$  is any tree with five edges.

**Lemma 1.3.16** *For positive integers  $m$  and  $n$ , the graph  $K_{5m,n}$  has a  $B$ -decomposition and a  $C$ -decomposition if  $n \geq 2$ , a  $D$ -decomposition and an  $E$ -decomposition if  $n \geq 3$ , and a  $P_6$ -decomposition if  $n \geq 4$ .*

**Proof.** Let  $m$  and  $n$  be positive integers,  $n \geq 2$ . We first consider  $B$ . It suffices to show that  $K_{5,2}$  and  $K_{5,3}$  have  $B$ -decompositions. For  $K_{5,2}$ , label the vertices with a  $(\mathbb{Z}_5, \mathbb{Z}_2)$ -labeling. The following trees form a  $B$ -decomposition of  $K_{5,2}$  (see Figure 1.12).

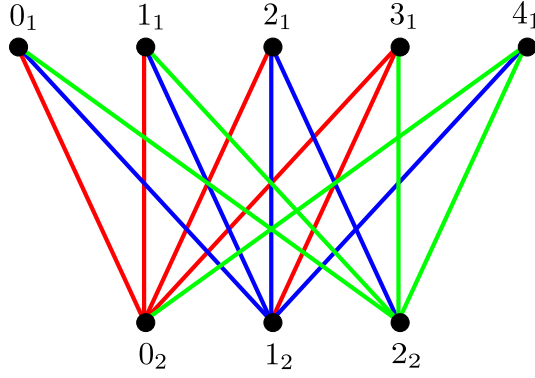
$$(0_2; 0_1, 1_1, 2_1, 3_1 - 1_2), (1_2; 0_1, 1_1, 2_1, 4_1 - 0_2)$$

Figure 1.12: A  $B$ -decomposition of  $K_{5,2}$ 

For  $K_{5,3}$ , label the vertices with a  $(\mathbb{Z}_5, \mathbb{Z}_3)$ -labeling. The following trees form a  $B$ -decomposition of  $K_{5,3}$  (see Figure 1.13).

$$(0_2; 0_1, 1_1, 2_1, 3_1 - 1_2), (1_2; 0_1, 1_1, 4_1, 2_1 - 2_2), (2_2; 0_1, 1_1, 3_1, 4_1 - 0_2)$$

Now we consider  $C$ . Again, it suffices to show that the graphs  $K_{5,2}$  and  $K_{5,3}$  have

Figure 1.13: A  $B$ -decomposition of  $K_{5,3}$ 

$C$ -decompositions.

For  $K_{5,2}$ , label the vertices with a  $(\mathbb{Z}_5, \mathbb{Z}_2)$ -labeling. The following trees form a  $C$ -decomposition of  $K_{5,2}$ .

$$(0_2; 0_1, 1_1, 2_1 - 1_2 - 3_1), (1_2; 0_1, 1_1, 4_1 - 0_2 - 3_1)$$

For  $K_{5,3}$ , label the vertices with a  $(\mathbb{Z}_5, \mathbb{Z}_3)$ -labeling. The following trees form a  $C$ -decomposition of  $K_{5,3}$ .

$$(0_2; 0_1, 1_1, 2_1 - 1_2 - 3_1), (1_2; 0_1, 1_1, 4_1 - 2_2 - 2_1), (2_2; 0_1, 1_1, 3_1 - 0_2 - 4_1)$$

Now let  $n \geq 3$ . First we consider  $D$ . It suffices to prove that the graphs  $K_{5,3}$ ,  $K_{5,4}$ , and  $K_{5,5}$  have  $D$ -decompositions.

For  $K_{5,3}$ , label the vertices with a  $(\mathbb{Z}_5, \mathbb{Z}_3)$ -labeling. The following graphs form a  $D$ -decomposition of  $K_{5,3}$ .

$$(0_1; 2_2, 0_2, 1_2 - 1_1, 2_1), (3_1; 0_2, 1_2, 2_2 - 1_1, 2_1), (4_1; 1_2, 0_2, 2_2 - 2_1, 1_1)$$

For  $K_{5,4}$ , label the vertices with a  $(\mathbb{Z}_5, \mathbb{Z}_4)$ -labeling. The following graphs form a  $D$ -decomposition of  $K_{5,4}$ .

$$(0_2; 0_1, 1_1, 2_1 - 2_2, 1_2), (1_2; 0_1, 3_1, 4_1 - 2_2, 3_2), \\ (2_2; 0_1, 2_1, 4_1 - 3_2, 0_2), (3_2; 0_1, 1_1, 3_1 - 1_2, 0_2)$$

For  $K_{5,5}$ , label the vertices with a  $(\mathbb{Z}_5, \mathbb{Z}_5)$ -labeling. The following graphs form a  $D$ -decomposition of  $K_{5,5}$ .

$$(3_1; 2_2, 0_2, 1_2 - 0_1, 2_1), (4_1; 0_2, 1_2, 2_2 - 0_1, 2_1), (0_1; 2_2, 3_2, 4_2 - 3_1, 4_1), \\ (1_1; 2_2, 0_2, 3_2 - 2_1, 4_1), (4_2; 3_1, 1_1, 2_1 - 1_2, 3_2)$$

Now consider  $E$ . It suffices to prove the existence of an  $E$ -decomposition of the graphs  $K_{5,3}$ ,  $K_{5,4}$ , and  $K_{5,5}$ .

For  $K_{5,3}$ , label the vertices with a  $(\mathbb{Z}_5, \mathbb{Z}_3)$ -labeling. The following trees form an  $E$ -decomposition of the graph  $K_{5,3}$ .

$$(0_2 - 0_1, 1_1; 2_1 - 1_2, 2_2), (1_2 - 0_1, 1_1; 3_1 - 0_2, 2_2), (2_2 - 0_1, 1_1; 4_1 - 0_2, 1_2)$$

For  $K_{5,4}$ , label the vertices with a  $(\mathbb{Z}_5, \mathbb{Z}_4)$ -labeling. The following trees form an



$E$ -decomposition of the graph  $K_{5,4}$ .

$$(0_2 - 1_1, 4_1; 2_1 - 1_2, 2_2), (1_2 - 1_1, 3_1; 4_1 - 2_2, 3_2), \\ (2_2 - 0_1, 1_1; 3_1 - 0_2, 3_2), (3_2 - 1_1, 2_1; 0_1 - 0_2, 1_2)$$

For  $K_{5,5}$ , label the vertices with a  $(\mathbb{Z}_5, \mathbb{Z}_5)$ -labeling. The following trees form an  $E$ -decomposition of the graph  $K_{5,5}$ . Note that the addition is taken modulo 5.

$$(i_1 - (i + 1)_2, (i + 2)_2; i_2 - (i + 1)_1, (i + 2)_1), i \in \mathbb{Z}_5$$

Finally, let  $n \geq 4$ . Parker proved that there exist  $P_6$ -decompositions of  $K_{5,4}$ ,  $K_{5,5}$ ,  $K_{5,6}$ , and  $K_{5,7}$  [32]. Therefore, for any  $n \geq 4$ , the graph  $K_{n,5}$  has a  $P_6$ -decomposition. ■

**Lemma 1.3.17** *If  $n \geq 7$  is an integer,  $T$  any tree with five edges, and  $K_n$  has a  $T$ -packing ( $T$ -covering) with the leave (excess) graph  $H$ , then  $K_{n+5}$  has a  $T$ -packing ( $T$ -covering) with the leave (excess) graph  $H$ . Furthermore, this statement is true if  $n = 6$  and  $T$  is any of  $B$ ,  $C$ ,  $D$ , or  $E$ , or if  $n = 5$  and  $T$  is either of  $B$  or  $C$ .*

**Proof.** Case 1.  $n \geq 5, T = B$

Let  $R$  be a  $B$ -packing of  $K_n$  with the leave graph  $H$ . Write  $K_{n+5} = K_n \vee K_5$ . Label the vertices of  $K_n \vee K_5$  with a  $(\mathbb{Z}_n, \mathbb{Z}_5)$ -labeling. The set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2, 4_2\}$ , the edges between these two sets, and the edges within the second set form a graph  $3K_1 \vee K_5$ . The following trees form a  $B$ -decomposition,  $S$ , of  $3K_1 \vee K_5$ .

$$(i_2; 0_1, 1_1, (i + 1)_2, (i + 2)_2 - 2_1), i \in \mathbb{Z}_5$$

Now, the set of vertices  $\{3_1, 4_1, 5_1, \dots, (n - 1)_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2, 4_2\}$ , and the edges between these two sets form a complete bipartite graph  $K_{5, n-3}$ , which has a  $B$ -decomposition,  $U$ , by Lemma 1.3.16. Therefore,  $R \cup S \cup U$  forms a  $B$ -packing of  $K_{n+5}$  with the leave graph  $H$ .

Case 2.  $n \geq 5, T = C$

Let  $R$  be a  $C$ -packing of  $K_n$  with the leave graph  $H$ . Write  $K_{n+5} = K_n \vee K_5$ . Label the vertices of  $K_n \vee K_5$  with a  $(\mathbb{Z}_n, \mathbb{Z}_5)$ -labeling. The set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2, 4_2\}$ , the edges between these two sets, and the edges within

the second set form a graph  $K_5 \vee 3K_1$ . The following trees form a  $C$ -decomposition,  $S$ , of the graph  $K_5 \vee 3K_1$ .

$$(0_2; 1_2, 2_2, 0_1 - 3_2 - 1_1), (1_2; 2_2, 3_2, 1_1 - 4_2 - 2_1), (2_2; 3_2, 4_2, 2_1 - 0_2 - 1_1), \\ (3_2; 4_2, 0_2, 2_1 - 1_2 - 0_1), (4_2; 0_2, 1_2, 0_1 - 2_2 - 1_1)$$

Moreover, the set of vertices  $\{3_1, 4_1, 5_1, \dots, (n-1)_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2, 4_2\}$ , and the edges between these two sets form a complete bipartite graph  $K_{5, n-3}$ , which has a  $C$ -decomposition,  $U$ , by Lemma 1.3.16. Therefore,  $R \cup S \cup U$  forms a  $C$ -packing with the leave graph  $H$  for  $K_{n+5}$ .

Case 3.  $n \geq 6, T = D$

Let  $R$  be a  $D$ -packing of  $K_n$  with the leave graph  $H$ . Write  $K_{n+5} = K_n \vee K_5$ . Label the vertices of  $K_n \vee K_5$  with a  $(\mathbb{Z}_n, \mathbb{Z}_5)$ -labeling. The set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2, 4_2\}$ , the edges between these two sets, and the edges within the second set form a graph  $K_5 \vee 3K_1$ . The following graphs form a  $D$ -decomposition,  $S$ , of  $K_5 \vee 3K_1$ .

$$(i_2; 0_1, (i+1)_2, (i+2)_2 - 1_1, 2_1), i \in \mathbb{Z}_5$$

The complete bipartite graph  $K_{5, n-3}$  with partite sets  $\{3_1, 4_1, 5_1, \dots, (n-1)_1\}$  and  $\{0_2, 1_2, 2_2, 3_2, 4_2\}$ , has a  $D$ -decomposition,  $U$ , by Lemma 1.3.16. Therefore,  $R \cup S \cup U$  forms a  $D$ -packing of  $K_{n+5}$  with the leave graph  $H$ .

Case 4.  $n \geq 6, T = E$

Let  $R$  be an  $E$ -packing of  $K_n$  with the leave graph  $H$ . Write  $K_{n+5} = K_n \vee K_5$ . Label the vertices of  $K_n \vee K_5$  with a  $(\mathbb{Z}_n, \mathbb{Z}_5)$ -labeling. The set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2, 4_2\}$ , the edges between these sets, and the edges within the second set, form a graph  $K_5 \vee 3K_1$ . The following trees form an  $E$  decomposition of the graph  $K_5 \vee 3K_1$ .

$$(i_2 - 0_1, (i+2)_2; (i+1)_2 - 1_1, 2_1), i \in \mathbb{Z}_5$$

Since  $n \geq 6$ , the complete bipartite graph with partite sets  $\{3_1, 4_1, 5_1, \dots, (n-1)_1\}$  and  $\{0_2, 1_2, 2_2, 3_2, 4_2\}$  has an  $E$ -decomposition,  $S$ , by Lemma 1.3.16. Therefore,  $R \cup S$  forms an  $E$ -packing of  $K_{n+5}$  with the leave graph  $H$ .

Case 5.  $n \geq 7, T = P_6$

Let  $R$  be a  $P_6$ -packing of  $K_n$  with the leave graph  $H$ . Write  $K_{n+5} = K_n \vee K_5$ . Label the vertices of  $K_n \vee K_5$  with a  $(\mathbb{Z}_n, \mathbb{Z}_5)$ -labeling. The set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2, 4_2\}$ , the edges between these sets, and the edges within the second set, form a graph  $K_5 \vee 3K_1$ . The following paths form a  $P_6$ -decomposition,  $S$ , of  $K_5 \vee 3K_1$ .

$$\begin{aligned} & (1_1, 1_2, 0_1, 0_2, 2_2, 3_2), (0_1, 2_2, 2_1, 1_2, 3_2, 4_2), (2_1, 3_2, 1_1, 2_2, 4_2, 0_2), \\ & (2_1, 0_2, 1_1, 4_2, 1_2, 2_2), (2_1, 4_2, 0_1, 3_2, 0_2, 1_2) \end{aligned}$$

By Lemma 1.3.16, the complete bipartite graph with partite sets  $\{3_1, 4_1, 5_1, \dots, (n-1)_1\}$  and  $\{0_2, 1_2, 2_2, 3_2, 4_2\}$  has a  $P_6$ -decomposition,  $U$ . Consequently,  $R \cup S \cup U$  forms a  $P_6$ -packing of  $K_{n+5}$  with the leave graph  $H$ .

The proof of the covering case uses a similar argument. ■

Now, we see an example that shows how we can reduce the problem of finding a maximum  $A$ -packing of the complete graph  $K_n$  for any integer  $n$  with a possible leave graph into the problem of finding a maximum packing of the complete graph  $K_m$  for an integer  $m \leq 15$ , using the lemmas in this chapter. Suppose we desire to find the leave graph  $3K_2$  in a maximum  $A$ -packing of  $K_{822}$ . Write  $K_{822} = K_{808} \vee K_{14}$ . By Theorem 1.3.1, the graph  $K_{808}$  has an  $A$ -decomposition,  $R$ . Also the graph  $K_{808,14}$  has an  $A$ -decomposition,  $S$ , by Lemma 1.3.14. Therefore, we have reduced the problem to finding a maximum  $A$ -packing,  $T$ , of the small graph  $K_{14}$  with the leave graph  $3K_2$ , such that  $R \cup S \cup T$  forms a maximum  $A$ -packing of  $K_{822}$  with the leave graph  $3K_2$ .

Generally, for any tree  $T$  with four edges and any integer  $n$ , we have  $n = 8k + k'$  for some positive integer  $k$  and integer  $k'$ , with  $0 \leq k' \leq 7$ . We write  $K_n = K_{n-(k'+8)} \vee K_{k'+8}$ , find a maximum  $T$ -packing of  $K_{k'+8}$  with the desired leave graph, and use Theorem 1.3.1 and Lemma 1.3.14 to find a maximum  $A$ -packing of  $K_n$  with that leave graph. In Chapters 2 and 3, we see how to achieve any desired leave graph in  $T$ -packings of  $K_{k'+8}$  for any tree  $T$  with four edges and any integer  $0 \leq k' \leq 7$ . For any tree  $T$  with five edges and any integer  $n$ , we use a similar method to achieve all leave and excess graphs in  $T$ -packings and  $T$ -coverings of the complete graph  $K_n$ .

## Chapter 2

# The Spectrum of Leave and Excess Graphs for Stars with up to Five Edges

This chapter will discuss different leave and excess graphs in packings and coverings of the complete graph with stars that have up to five edges.

### 2.1 The Spectrum of Leave and Excess Graphs for Stars with up to Three Edges

In this section, we solve the spectrum of leave and excess graphs for stars with up to three edges.

Since 1-stars consist of only one edge, for any integer  $n$ ,  $K_n$  has an  $S_1$ -decomposition.

For 2-stars,  $K_n$  has an  $S_2$ -decomposition if  $n$  is even [8]. If  $n$  is odd, then the leave (excess) graph in a maximum  $S_2$ -packing (minimum covering) of  $K_n$  has at most one edge, which has been achieved by Roditty [37].

For 3-stars,  $K_n$  has an  $S_3$ -decomposition if  $n \equiv 0$  or  $1 \pmod{3}$ , and  $n \neq 3, 4$ . For  $n \equiv 2 \pmod{3}$ ,  $n \geq 5$ , the leave graph in a maximum  $S_3$ -packing of  $K_n$  has one edge, which has been achieved by Roditty [37]. Also for  $n \equiv 2 \pmod{3}$ ,  $n \geq 6$ , the excess graph in a minimum  $S_3$ -covering of  $K_n$  has two edges. Hence, the possible

excess graphs are  $P_3$ ,  $2K_2$ , and  $K_2^2$ . We prove that all these possible excess graphs are achievable. By Lemma 1.3.10, it suffices to consider  $n = 8$ .

The excess graph  $P_3$  is achieved by Roditty [37]. In order to obtain the excess graph  $K_2^2$ , write  $K_8 = K_5 \vee K_3$ . Label the vertices of  $K_5 \vee K_3$  with a  $(\mathbb{Z}_5, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.2,  $K_5$  has an  $S_3$ -packing,  $R$ , with a single edge, say  $\{3_1, 4_1\}$  as the leave graph. Moreover, the following stars form a minimum  $S_3$ -covering,  $S$ , for the remaining graph with the edges  $\{0_2, 1_2\}$  used twice, as the excess graph.

$$(0_2; 1_2, 0_1, 1_1), (0_2; 1_2, 2_1, 3_1), (1_2; 0_2, 0_1, 1_1), (1_2; 2_2, 2_1, 3_1), \\ (2_2; 0_2, 0_1, 1_1), (2_2; 2_1, 3_1, 4_1), (4_1; 3_1, 0_2, 1_2)$$

Therefore,  $R \cup S$  forms a minimum  $S_3$ -covering of  $K_8$  with the excess graph  $K_2^2$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used twice. Now, substituting the stars  $(0_2; 0_1, 1_1, 2_1)$  and  $(1_2; 0_1, 1_1, 3_1)$  for  $(0_2; 1_2, 0_1, 1_1)$  and  $(1_2; 0_2, 0_1, 1_1)$  respectively in  $R \cup S$  will result in a minimum  $S_3$ -covering of  $K_8$  with the excess graph  $2K_2$ . The edges of the excess graph are  $\{0_2, 2_1\}$  and  $\{1_2, 3_1\}$ .

## 2.2 The Spectrum of Leave and Excess Graphs for 4-stars

In this section, we find a corresponding maximum packing and minimum covering of the complete graph with 4-stars for every possible leave and excess graph.

### 2.2.1 The Spectrum of Leave Graphs for 4-stars

**Theorem 2.2.1** *Let  $n \geq 7$  be an integer and let the leave graph in a maximum packing of the complete graph  $K_n$  with 4-stars have  $i$  edges. For any graph  $H$  with  $i$  edges there exists a maximum packing of  $K_n$  with 4-stars such that the leave graph is isomorphic to  $H$ .*

**Proof.** By Theorem 1.3.1,  $K_n$  has an  $S_4$ -decomposition for  $n \equiv 0$  or  $1 \pmod{8}$ . We show that for the remaining cases we have maximum packings with all the possible leave graphs.

Case 1.  $n \equiv 2$  or  $7 \pmod{8}$

By Theorem 1.3.2, the leave graph is a single edge and the proof is complete in this case.

Case 2.  $n \equiv 3 \pmod{8}$

In this case, the leave graph has three edges. The non-isomorphic possible leave graphs are  $S_3$ ,  $K_3$ ,  $P_3 + K_2$ ,  $P_4$ , and  $3K_2$ .

In order to get an  $S_3$  as the leave graph, write  $K_n = K_{n-3} \vee K_3$ . Since  $n \equiv 3 \pmod{8}$ , we have  $n - 3 \equiv 0 \pmod{8}$  and hence  $K_{n-3}$  has an  $S_4$ -decomposition,  $R$ , by Theorem 1.3.1. Label the vertices of  $K_{n-3} \vee K_3$  with a  $(\mathbb{Z}_{n-3}, \mathbb{Z}_3)$ -labeling. Now, the set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges between these two sets of vertices, and the edges within the second set form a  $K_3 \vee 3K_1$ . By Lemma 1.3.6,  $K_3 \vee 3K_1$  has an  $S_4$ -decomposition,  $S$ . Now, the set of vertices  $\{3_1, 4_1, \dots, (n-5)_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , and the edges between these two sets of vertices form a complete bipartite graph which has one part of size a multiple of 4. Therefore, by Lemma 1.3.5 this complete bipartite graph has an  $S_4$ -decomposition,  $T$ . Hence,  $R \cup S \cup T$  forms a maximum packing of  $K_n$  with 4-stars with the 3-star  $((n-4)_1; 0_2, 1_2, 2_2)$  as the leave graph.

In order to obtain  $3K_2$  as the leave graph, again write  $K_n = K_{n-3} \vee K_3$ , if  $n \geq 19$ . Label the vertices as above and let  $R$  and  $S$  be the same decompositions. Now, the set of vertices  $\{3_1, 4_1, \dots, (n-9)_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , and the edges between these two sets of vertices form a complete bipartite graph with one part of size a multiple of 4. Hence, by Lemma 1.3.5, this complete bipartite graph has an  $S_4$ -decomposition,  $T$ . Now, the set of vertices  $\{(n-8)_1, (n-7)_1, (n-6)_1, (n-5)_1, (n-4)_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , and the edges between these two sets of vertices form a  $K_{3,5}$ . By Lemma 1.3.4,  $K_{3,5}$  has a maximum packing,  $Q$ , with the leave graph  $3K_2$ . Hence,  $R \cup S \cup T \cup Q$  forms a maximum packing of  $K_n$  with 4-stars with the leave graph  $3K_2$ .

If  $n = 11$ , in order to achieve  $3K_2$  as the leave graph, write  $K_{11} = K_8 \vee K_3$ . Label the vertices of  $K_8 \vee K_3$  with a  $(\mathbb{Z}_8, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.1,  $K_8$  has an  $S_4$ -decomposition,  $R$ . The set of vertices  $\{0_1, 1_2, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges within the latter set, and the edges between these two sets, form a graph  $K_3 \vee 3K_1$ , which has an  $S_4$ -decomposition,  $S$ , by Lemma 1.3.6. Moreover, the set of

vertices  $\{3_1, 4_1, 5_1, 6_1, 7_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , and the edges between these two sets, form a complete bipartite graph  $K_{3,5}$ , which has a maximum  $S_4$ -packing,  $T$ , with the leave graph  $3K_2$ , by Lemma 1.3.4. Therefore,  $R \cup S \cup T$  is a maximum  $S_4$ -packing of  $K_8$  with the leave graph  $3K_2$ .

Now, to achieve  $K_3$  as the leave graph, write  $K_n = K_{n-1} \vee K_1$ . Label the vertices of  $K_{n-1}$  with the elements of  $\mathbb{Z}_{n-1}$  and the single vertex of  $K_1$  with  $\infty$ . Since  $n \equiv 3 \pmod{8}$ , by Theorem 1.3.2,  $K_{n-1}$  has a maximum  $S_4$ -packing,  $R$ , with the edge  $\{n-3, n-2\}$  as the leave graph. Moreover, the set of vertices  $\{0, 1, \dots, n-4\}$ , the set of vertex  $\{\infty\}$ , and the edges between these two sets, form a graph  $K_{n-3,1}$ , which has an  $S_4$ -decomposition,  $S$ , by Lemma 1.3.5. Therefore,  $R \cup S$  forms a maximum  $S_4$ -packing of  $K_n$  with the leave graph  $K_3$ . The edges of the leave graph are  $\{n-3, n-2\}$ ,  $\{n-2, \infty\}$ , and  $\{\infty, n-3\}$ . Figure 2.1 illustrates the last step in achieving a maximum  $S_4$ -packing of  $K_{11}$  with the leave graph  $K_3$ . Each thick line connected to an oval takes the place of a 4-star.

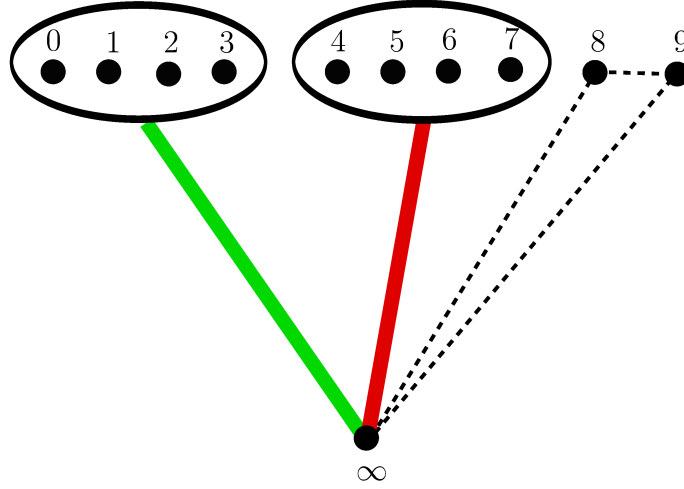


Figure 2.1: The last step in achieving a maximum  $S_4$ -packing of  $K_{11}$  with the leave graph  $K_3$

In order to obtain  $P_4$  as the leave graph, again write  $K_n = K_{n-1} \vee K_1$  and label the vertices in the same way, and let  $R$  be the same maximum packing with the edge  $\{n-3, n-2\}$  as the leave graph. The set of vertices  $\{1, 2, \dots, n-3\}$ , the set of vertex  $\{\infty\}$ , and the edges between these two sets of vertices form a complete bipartite graph with one part of size a multiple of 4. Hence, by Lemma 1.3.5, this complete bipartite graph has an  $S_4$ -decomposition,  $S$ . Therefore,  $R \cup S$  forms a maximum  $S_4$ -packing of

$K_n$  where the three edges  $\{n-3, n-2\}$ ,  $\{n-2, \infty\}$ , and  $\{\infty, 0\}$  are left, which form a  $P_4$ . Figure 2.2 illustrates the last step in achieving a maximum  $S_4$ -packing of  $K_{11}$  with the leave graph  $P_4$ . Each thick line connected to an oval takes the place of a 4-star.

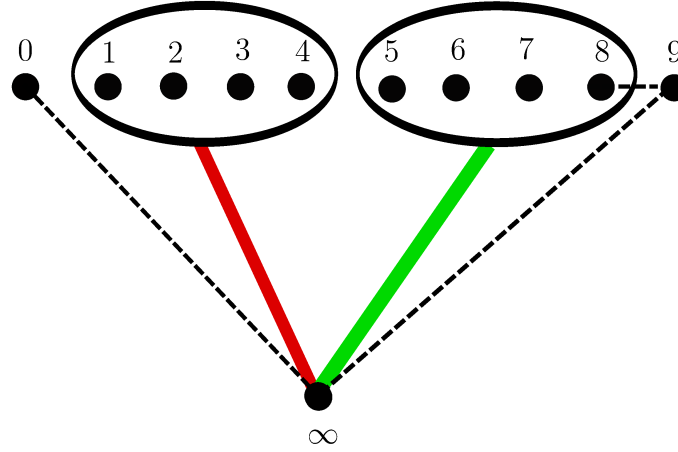


Figure 2.2: The last step in achieving a maximum  $S_4$ -packing of  $K_{11}$  with the leave graph  $P_4$

Finally, to achieve  $P_3 + K_2$  as the leave graph, again write  $K_n = K_{n-1} \vee K_1$ , label the vertices the same way, and let  $R$  be the same maximum packing with the same leave graph. The set of vertices  $\{2, 3, \dots, n-2\}$ , the set of vertex  $\{\infty\}$ , and the edges between these two sets of vertices form a complete bipartite graph with one part of size a multiple of 4. Hence, by Lemma 1.3.5, this complete bipartite graph has an  $S_4$ -decomposition,  $S$ . Therefore,  $R \cup S$  forms a maximum  $S_4$ -packing of  $K_n$  where the three edges  $\{n-3, n-2\}$ ,  $\{0, \infty\}$ , and  $\{\infty, 1\}$  are left which form an  $P_3 + K_2$  (see Figure 2.3, in which each thick line connected to an oval takes the place of a 4-star). Figure 2.3 illustrates the last step in achieving a maximum  $S_4$ -packing of  $K_{11}$  with the leave graph  $P_3 + K_2$ . Each thick line connected to an oval takes the place of a 4-star. This completes the proof in this case. Case 3.  $n \equiv 4 \pmod{8}$

By Theorem 1.3.2 the leave graph has two edges in this case. Hence, the possible leave graphs are  $P_3$  and  $2K_2$ . In order to obtain  $P_3$  as the leave graph, write  $K_n = K_{n-1} \vee K_1$ . Label the vertices of  $K_{n-1}$  with the elements of  $\mathbb{Z}_{n-1}$  and the single vertex of  $K_1$  with  $\infty$ . Since  $n \equiv 4 \pmod{8}$ ,  $K_{n-1}$  has a maximum  $S_4$ -packing,  $R$ , with an  $S_3$  as the leave graph as stated in Case 2. Let the edges in this leave graph be



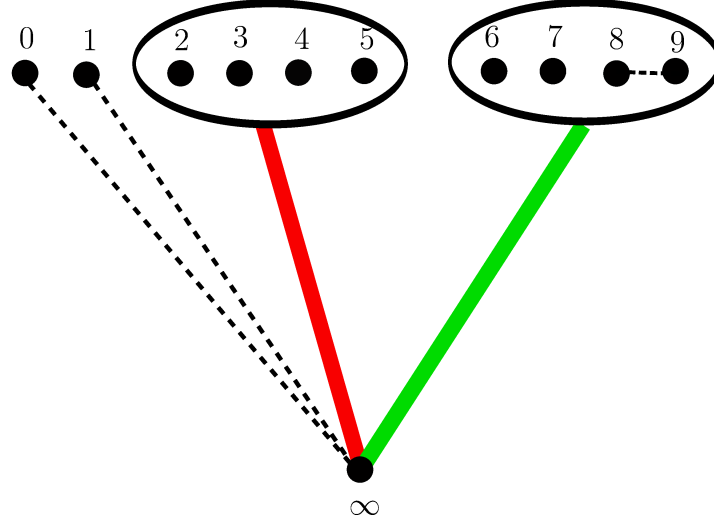


Figure 2.3: The last step in achieving a maximum  $S_4$ -packing of  $K_{11}$  with the leave graph  $P_3 + K_2$

$\{n-2, n-3\}$ ,  $\{n-2, n-4\}$ , and  $\{n-2, n-5\}$ . Now, the set of vertices  $\{0, 1, \dots, n-5\}$ , the set of vertex  $\{\infty\}$ , and the edges between these two sets of vertices form a complete bipartite graph with one part of size a multiple of 4. Hence, by Lemma 1.3.5, this complete bipartite graph has an  $S_4$ -decomposition,  $S$ . Therefore, we are left with the edges  $\{n-2, n-3\}$ ,  $\{n-2, n-4\}$ ,  $\{n-2, n-5\}$ ,  $\{\infty, n-4\}$ ,  $\{\infty, n-3\}$ , and  $\{\infty, n-2\}$ . Therefore,  $R \cup S \cup \{(n-2; n-3, n-4, n-5, \infty)\}$  forms a maximum  $S_4$ -packing of  $K_n$  where the two edges  $\{\infty, n-3\}$  and  $\{\infty, n-4\}$  are left, which form a  $P_3$ .

In order to achieve  $2K_2$  as the leave graph, write  $K_n = K_{n-4} \vee K_4$ . Label the vertices of  $K_{n-4} \vee K_4$  with a  $(\mathbb{Z}_{n-4}, \mathbb{Z}_4)$ -labeling. Since  $n \equiv 4 \pmod{8}$ ,  $K_{n-4}$  has an  $S_4$ -decomposition,  $R$ . The set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , the edges between these two sets, and the edges within the second set form a graph which we call  $H$ . The following 4-stars form a maximum  $S_4$ -packing of  $H$ ,  $S$ , with the edges  $\{0_2, 2_2\}$  and  $\{1_2, 3_2\}$  as the leave graph, which form a  $2K_2$ .

$$(i_2; (i+1)_2, 0_1, 1_1, 2_1); i \in \mathbb{Z}_4.$$

Now, the set of vertices  $\{3_1, 4_1, \dots, (n-5)_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , and the edges between these two sets of vertices form a complete bipartite graph with one part of size a multiple of 4. Hence, by Lemma 1.3.5, this complete bipartite graph

has an  $S_4$ -decomposition,  $T$ . Therefore,  $R \cup S \cup T$  forms a maximum  $S_4$ -packing of  $K_n$  where the edges  $\{0_2, 2_2\}$  and  $\{1_2, 3_2\}$  are left which form a  $2K_2$ . This completes the proof in this case.

Case 4.  $n \equiv 5 \pmod{8}$

In this case, again by Theorem 1.3.2, the leave graph has two edges. Write  $K_n = K_{n-1} \vee K_1$ . Since  $n \equiv 5 \pmod{8}$ , we have both of the possible leave graphs for  $K_{n-1}$  by Case 3. Let  $H$  be one of the leave graphs. Since  $n-1$  is a multiple of 4, by Lemma 1.3.5,  $K_{n-1,1}$  has an  $S_4$ -decomposition. So, the leave graph is  $H$  and the proof is completed in this case.

Case 5.  $n \equiv 6 \pmod{8}$

By Theorem 1.3.2, the leave graph has three edges in this case. Write  $K_n = K_{n-3} \vee K_3$ . Since  $n \equiv 6 \pmod{8}$ , we have all the possible leave graphs of  $S_4$ -packings of in  $K_{n-3}$  from Case 2. Let  $H$  be one of those leave graphs and  $R$  be the corresponding packing. Label the vertices of  $K_{n-3} \vee K_3$  with a  $(\mathbb{Z}_{n-3}, \mathbb{Z}_3)$ -labeling. The set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges between these two sets of vertices, and the edges between the vertices in the second set form a  $K_3 \vee 3K_1$ . By Lemma 1.3.4 this graph has an  $S_4$ -decomposition,  $S$ . Now, the set of vertices  $\{3_1, 4_1, \dots, (n-4)_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , and the edges between these two sets of vertices form a complete bipartite graph with one part of size  $n-6$ . Since  $n \equiv 6 \pmod{8}$ ,  $n-6$  is a multiple of 4 and hence, this complete bipartite graph has an  $S_4$ -decomposition,  $T$ , by Lemma 1.3.5. Therefore,  $R \cup S \cup T$  forms a maximum  $S_4$ -packing of  $K_n$  with 4-stars with the leave graph  $H$  and this completes the proof in this case.

■

Note that for  $n = 6$  the only possible leave graph is  $K_3$ , which shows that the condition  $n \geq 7$  in Theorem 2.2.1 is necessary. In order to prove that for  $n = 6$  the only possible leave graph is  $K_3$ , label the vertices of  $K_6$  with the elements of  $\mathbb{Z}_6$ . Any maximum packing contains 3 stars. Without loss of generality we assume the first star to be  $(0; 1, 2, 3, 4)$ . We have two options for the next star center.

Assume we choose vertex 5 as the center of our next star. We can choose the leaves of the star to be the vertices 1, 2, 3, and 4 or choose one of the leaves to be the vertex 0 and the others to be three of the vertices 1, 2, 3, and 4. The first choice is impossible since every vertex will have degree at least two and we cannot add the

third star. Hence, without loss of generality assume the second star to be  $(5; 0, 1, 2, 3)$  and we have to choose  $(4; 1, 2, 3, 5)$  as the third star and the leave graph will be the triangle with the edges  $\{1, 2\}$ ,  $\{2, 3\}$ , and  $\{3, 1\}$ .

Now, assume we choose one of the vertices of degree one to be the center of our second star. Without loss of generality we can take  $(1; 2, 3, 4, 5)$  as the second star. Hence, the only possibility for the third star will be  $(5; 0, 2, 3, 4)$  which gives a triangle with the edges  $\{2, 3\}$ ,  $\{3, 4\}$ , and  $\{4, 2\}$  as the leave graph which completes the proof.

### 2.2.2 The Spectrum of Excess Graphs for 4-stars

In the previous subsection we illustrated how we can achieve all the possible leave graphs in an  $S_4$ -packing of  $K_n$ . Now, we show that we can obtain every possible excess graph in a minimum  $S_4$ -covering of  $K_n$  as well.

**Theorem 2.2.2** *Let  $n \geq 8$  be an integer and let the excess graph in a minimum  $S_4$ -covering of the complete graph  $K_n$  have  $i$  edges. For any graph  $H$  with  $i$  edges there exists a minimum  $S_4$ -covering of  $K_n$  such that the excess graph is isomorphic to  $H$ .*

**Proof.** Again we know that for  $n \equiv 0$  or  $1 \pmod{8}$ ,  $K_n$  has an  $S_4$ -decomposition. We show that for the remaining cases we have minimum coverings with all the possible excess graphs.

Case 1.  $n \equiv 2 \pmod{8}$

By Theorem 1.3.3, the excess graph has three edges in this case. The possible excess graphs with three edges are  $S_3$ ,  $K_3$ ,  $P_4$ ,  $P_3 + K_2$ ,  $3K_2$ ,  $K_2^3$ ,  $K_2^2 + K_2$ , and  $F$ , where  $F$  is the graph  $K_2^2$  with an edge attached to one of its vertices.

We can obtain the excess graph  $S_3$  from a maximum  $S_4$ -packing of  $K_n$  with the leave graphs  $K_2$ , adding a 4-star which has the leave graph of the packing as an edge.

For the excess graphs  $K_3$ ,  $P_4$ ,  $P_3 + K_2$ , and  $3K_2$ , we use the following construction. Write  $K_n = K_{n-3} \vee K_3$ . Label the vertices of  $K_{n-3} \vee K_3$  with a  $(\mathbb{Z}_{n-3}, \mathbb{Z}_3)$ -labeling. Since  $n \equiv 2 \pmod{8}$ , by Case 6 in the proof of Theorem 2.2.1,  $K_{n-3}$  has an  $S_4$ -packing,  $R$ , with a single edge as the leave graph. Let  $\{(n-5)_1, (n-4)_1\}$  be that single edge. Consider the set of vertices  $\{0_1, 1_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges between these two sets, and the edges within the second set. The following 4-stars form a

minimum covering called  $S$  with the triangle formed by the edges  $\{0_2, 1_2\}$ ,  $\{0_2, 2_2\}$ , and  $\{1_2, 2_2\}$  as the excess graph.

$$(0_2; 1_2, 2_2, 0_1, 1_1), (1_2; 0_2, 2_2, 0_1, 1_1), (2_2; 0_2, 1_2, 0_1, 1_1)$$

Now, consider the set of vertices  $\{2_1, 3_1, \dots, (n-5)_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , and the edges between these two sets. These form a complete bipartite graph with one part of size a multiple of 4 since  $n \equiv 2 \pmod{8}$ . Hence, by Lemma 1.3.5, this complete bipartite graph has an  $S_4$ -decomposition,  $T$ . Therefore,  $R \cup S \cup T \cup ((n-4)_1; (n-5)_1, 0_2, 1_2, 2_2)$  forms a minimum  $S_4$ -covering of  $K_n$  with a  $K_3$  as the excess graph. The edges of the excess graph are  $\{0_2, 1_2\}$ ,  $\{0_2, 2_2\}$ , and  $\{1_2, 2_2\}$  (see Figure 2.4, in which the thick line connected to an oval takes the place of a 4-star).

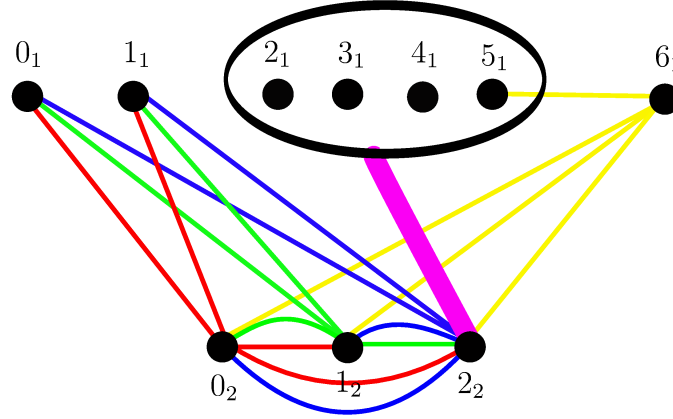


Figure 2.4: An  $S_4$ -covering of  $K_{10}$  with the excess graph  $K_3$

Consider the stars in the minimum covering above. Replacing the star  $(2_2; 0_2, 1_2, 0_1, 1_1)$  with  $(2_2; 0_2, (n-4)_1, 0_1, 1_1)$  gives the path  $P_4$  as the excess graph. In fact, the excess graph is the path  $((n-4)_1, 2_2, 0_2, 1_2)$ .

Replacement of the star  $(2_2; 0_2, 1_2, 0_1, 1_1)$  with  $(2_2; (n-4)_1, (n-5)_1, 0_1, 1_1)$  leads to the excess graph  $P_3 + K_2$ , with the edges  $\{0_2, 1_2\}$ ,  $\{(n-5)_1, 2_2\}$ , and  $\{(n-4)_1, 2_2\}$ .

If we replace the stars  $(0_2; 1_2, 2_2, 0_1, 1_1)$ ,  $(1_2; 0_2, 2_2, 0_1, 1_1)$ , and  $(2_2; 0_2, 1_2, 0_1, 1_1)$  with  $(0_2; 1_2, 2_1, 0_1, 1_1)$ ,  $(1_2; 3_1, 2_2, 0_1, 1_1)$ , and  $(2_2; 0_2, 4_1, 0_1, 1_1)$ , then the excess graph will be a  $3K_2$ , with the edges  $\{2_1, 0_2\}$ ,  $\{3_1, 1_2\}$ , and  $\{4_1, 2_2\}$ .

For the remaining possible excess graphs, we use the following construction. Again write  $K_n = K_{n-3} \vee K_3$ . Label the vertices of  $K_{n-3} \vee K_3$  with a  $(\mathbb{Z}_{n-3}, \mathbb{Z}_3)$ -labeling.

Since  $n \equiv 2 \pmod{8}$ , we have  $n - 3 \equiv 7 \pmod{8}$ . Hence, the leave graph in the  $S_4$ -packing of  $K_{n-3}$  has one edge by Theorem 1.3.2. Let  $R$  be a maximum  $S_4$ -packing of  $K_{n-3}$  with 4-stars and the single edge  $\{(n-5)_1, (n-4)_1\}$  be the corresponding leave graph. The following stars along with the ones in  $R$  form a minimum  $S_4$ -covering of  $K_n$  with 4-stars with the edges  $\{0_2, 1_2\}$ , used three times, as the excess graph, which is a  $K_2^3$ . Figure 2.5 illustrates the last step in achieving a minimum  $S_4$ -covering of  $K_{10}$  with the excess graph  $K_2^3$ . Each thick line connected to an oval takes the place of a 4-star.

$$\begin{aligned}
& (0_2; 1_2, i_1, (i+1)_1, (i+2)_1), i = 0 \text{ and } 3 \\
& (1_2; 0_2, i_1, (i+1)_1, (i+2)_1), i = 0 \text{ and } 3 \\
& (i_2; (4j+6)_1, (4j+7)_1, (4j+8)_1, (4j+9)_1), 0 \leq i \leq 1, 0 \leq j \leq \frac{n-14}{4}, i, j \in \mathbb{Z} \\
& (2_2; (4j)_1, (4j+1)_1, (4j+2)_1, (4j+3)_1), 0 \leq j \leq \frac{n-10}{4}, j \in \mathbb{Z} \\
& ((n-4)_1; (n-5)_1, 0_2, 1_2, 2_2), \\
& (2_2; 0_2, 1_2, (n-6)_1, (n-5)_1).
\end{aligned}$$

For  $n = 10$ , the same construction works, ignoring the third line of the above set of stars,  $R$ .

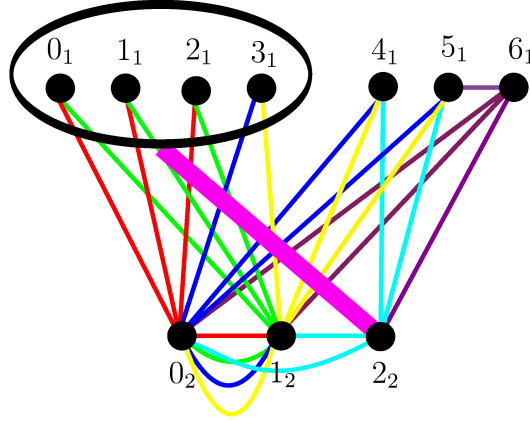


Figure 2.5: The last step in achieving a minimum  $S_4$ -covering of  $K_{10}$  with the excess graph  $K_2^3$

In the same covering as above, replace the star  $(0_2; 1_2, 0_1, 1_1, 2_1)$  with  $(0_2; 2_2, 0_1, 1_1, 2_1)$  to achieve the excess graph  $F$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$

used twice, and the edge  $\{0_2, 2_2\}$ .

Consider the covering with excess graph  $D$  and replace the stars  $(1_2; 0_2, 0_1, 1_1, 2_1)$  and  $(1_2; 0_2, 3_1, 4_1, 5_1)$  with  $(1_2; (n-4)_1, 0_1, 1_1, 2_1)$  and  $(1_2; (n-4)_1, 3_1, 4_1, 5_1)$  to give the excess graph  $K_2^2 + K_2$ . The edges of the excess graph are the edges  $\{(n-4)_1, 1_2\}$  used twice, and the edge  $\{0_2, 2_2\}$ . This proves the theorem in the first case.

Case 2.  $n \equiv 3$  or  $6 \pmod{8}$

By Theorem 1.3.3, the excess graph is a single edge and the proof is complete in this case.

Case 3.  $n \equiv 4 \pmod{8}$

Again by Theorem 1.3.3, the excess graph has two edges. The possible graphs with two edges are  $P_3$ ,  $2K_2$ , and  $K_2^2$ . The excess graph  $P_3$  is easily obtained from a maximum packing with the leave graph  $P_3$ .

In order to obtain the excess graph  $2K_2$  write  $K_n = K_{n-1} \vee K_1$ . Label the vertex  $K_1$  with  $\infty$  and the vertices of  $K_{n-1}$  with the elements of  $\mathbb{Z}_{n-1}$ . Since  $n \equiv 4 \pmod{8}$ , we have  $n-1 \equiv 3 \pmod{8}$  and hence, the excess graph of an  $S_4$ -covering of  $K_{n-1}$  has a single edge. Let that single edge be  $\{n-3, n-2\}$ . The following stars along with those in a minimum  $S_4$ -covering of  $K_{n-1}$  form a minimum  $S_4$ -covering for  $K_n$  with the excess graph  $2K_2$ . The edges of the excess graph are  $\{0, \infty\}$  and  $\{n-3, n-2\}$ .

$$(\infty; 4i, 4i+1, 4i+2, 4i+3), i \in \left\{0, 1, \dots, \frac{n-8}{4}\right\},$$

$$(\infty; 0, n-4, n-3, n-2).$$

The following construction gives the excess graph  $K_2^2$ . Write  $K_n = K_{n-3} \vee K_3$ . Since  $n \equiv 4 \pmod{8}$ ,  $K_{n-3}$  has an  $S_4$ -decomposition. Partition the vertices of  $K_{n-3}$  into a set of three vertices, a set of two vertices, and a set of  $n-8$  vertices. First, consider the set of three vertices. By Lemma 1.3.6,  $K_3 \vee 3K_1$  has an  $S_4$ -decomposition. Now, consider the set of  $n-8$  vertices. Since  $n \equiv 4 \pmod{8}$ ,  $n-8$  is a multiple of 4. Hence, by Lemma 1.3.5,  $K_{3, n-8}$  has an  $S_4$ -decomposition. Consider the two vertices left from the vertex partition of  $K_{n-3}$  and the vertices of  $K_3$ , and label them with a  $(\mathbb{Z}_2, \mathbb{Z}_3)$ -labeling. The following stars along with those in the decompositions of  $K_{n-3}$ ,  $K_3 \vee 3K_1$ , and  $K_{3, n-8}$  form a minimum  $S_4$ -covering of  $K_n$  with the edges  $\{0_1, 1_1\}$  used

twice, as the excess graph, which forms a  $K_2^2$ .

$$(0_1; 1_1, 0_2, 1_2, 2_2), (1_1; 0_1, 0_2, 1_2, 2_2).$$

Case 4.  $n \equiv 5 \pmod{8}$

By Theorem 1.3.3, the excess graph has two edges. Let  $H$  be one of the possible graphs with two edges. Write  $K_n = K_{n-1} \vee K_1$ . Since  $n \equiv 5 \pmod{8}$ , by Case 3,  $K_{n-1}$  has a minimum covering with the excess graph  $H$ . Since  $n-1$  is a multiple of 4,  $K_{1,n-1}$  has an  $S_4$ -decomposition by Lemma 1.3.5. Hence, the stars in the decomposition of  $K_{1,n-1}$  along with those in the minimum  $S_4$ -covering of  $K_{n-1}$  form a minimum  $S_4$ -covering of  $K_n$  with the excess graph  $H$ .

Case 5.  $n \equiv 7 \pmod{8}$

In this case, the excess graph has three edges. For  $n \geq 8$ , write  $K_n = K_{n-5} \vee K_5$ . Let  $H$  be any possible graph with three edges where multiple edges are allowed as well. Since  $n \equiv 7 \pmod{8}$ ,  $K_{n-5}$  has a minimum covering with excess graph  $H$  by Case 1. Partition the vertices of  $K_{n-5}$  into a set of six vertices and a set of  $n-11$  vertices. Consider the set of  $n-11$  vertices. Since  $n \equiv 7 \pmod{8}$ ,  $n-11$  is a multiple of 4. Hence, by Lemma 1.3.5,  $K_{5,n-11}$  has an  $S_4$ -decomposition. Now, consider the set of six vertices. By Lemma 1.3.6,  $K_5 \vee 6K_1$  has an  $S_4$ -decomposition. The stars in the decompositions of  $K_{5,n-11}$  and  $K_5 \vee 6K_1$  along with those in the minimum  $S_4$ -covering of  $K_{n-5}$  form a minimum  $S_4$ -covering of  $K_n$  with the excess graph  $H$ . ■

$n \pmod{8}$	Leave graph (for $n \geq 7$ )	Excess graph (for $n \geq 8$ )
0	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$
2	$K_2$	$S_3, K_3, P_4, 3K_2, P_3 + K_2, K_2^3, K_2^2 + K_2$ , and $F$
3	$S_3, K_3, P_4, 3K_2$ , and $P_3 + K_2$	$K_2$
4	$P_3$ and $2K_2$	$P_3, 2K_2$ , and $K_2^2$
5	$P_3$ and $2K_2$	$P_3, 2K_2$ , and $K_2^2$
6	$S_3, K_3, P_4, 3K_2$ , and $P_3 + K_2$	$K_2$
7	$K_2$	$S_3, K_3, P_4, 3K_2, P_3 + K_2, K_2^3, K_2^2 + K_2$ and $F$

Table 2.1: The spectrum of leave and excess graphs for 4-stars

Table 2.1 illustrates the spectrum of leave and excess graphs for 4-stars. In this table,  $F$  denotes the graph  $K_2^2$  with an edge attached to one of its vertices.

## 2.3 The Spectrum of Leave and Excess Graphs for 5-stars

In this section, we solve the spectrum problem for packing and covering for 5-stars.

### 2.3.1 The Spectrum of Leave Graphs for 5-stars

In 1986 Roditty solved the problem of packing the complete graph  $K_n$  with 5-stars. We prove that we can achieve all possible non-isomorphic leave graphs.

**Theorem 2.3.1** *Let  $n \geq 9$  be an integer and let the leave graph in a maximum  $S_5$ -packing of the complete graph  $K_n$  with 5-stars have  $i$  edges. For any graph  $H$  with  $i$  edges there exists a maximum  $S_5$ -packing of  $K_n$  such that the leave graph is isomorphic to  $H$ .*

**Proof.** The complete graph  $K_n$  has an  $S_5$ -decomposition for  $n \equiv 0, 1, 5,$  or  $6 \pmod{10}$  by Theorem 1.3.1. We show that for the remaining cases we have maximum packings with all the possible leave graphs.

By Corollary 1.3.9 and Lemma 1.3.10, the proof is complete for  $n \equiv 4$  and  $9 \pmod{10}$ . Now, by Lemma 1.3.10, we only need to prove the theorem for the cases when  $n \equiv 2$  and  $3 \pmod{10}$ . Again by Lemma 1.3.10, it suffices to prove the theorem for  $n = 12$  and  $n = 13$ .

Case 1.  $n = 12$

Write  $K_{12} = K_{10} \vee K_2$ . Label the vertices of  $K_{10} \vee K_2$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_2)$ -labeling. By Theorem 1.3.1,  $K_{10}$  has an  $S_5$ -decomposition,  $R$ . Now, the set of vertices  $\{0_1, 1_1, \dots, 9_1\}$ , the set of vertices  $\{0_2, 1_2\}$ , and the edges between these two sets form a complete bipartite graph with one part of size a multiple of 5. Hence, by Lemma 1.3.5, this complete bipartite graph has an  $S_5$ -decomposition,  $S$ . Now,  $R \cup S$  forms a maximum  $S_5$ -packing of  $K_{12}$  with the single edge  $\{0_2, 1_2\}$  as the leave graph.

Case 2.  $n = 13$

For this case, the leave graph has three edges by Theorem 1.3.2. Hence, the possible leave graphs are  $K_3$ ,  $S_3$ ,  $P_4$ ,  $3K_2$ , and  $P_3 + K_2$ . In order to obtain  $K_3$ , write  $K_{13} = K_{10} \vee K_3$ . The graph  $K_{10}$  has an  $S_5$ -decomposition,  $R$ , by Theorem



1.3.1. Moreover, 10 is a multiple of 5 and hence, by Lemma 1.3.5,  $K_{3,10}$  has an  $S_5$ -decomposition,  $S$ . Therefore,  $R \cup S$  forms a maximum  $S_5$ -packing of  $K_{13}$  with a  $K_3$  as the leave graph.

In order to obtain  $S_3$  as the leave graph, again write  $K_{13} = K_{10} \vee K_3$ . The graph  $K_{10}$  has an  $S_5$ -decomposition,  $R'$ , by Theorem 1.3.1. Label the vertices of  $K_{10} \vee K_3$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_3)$ -labeling. The set of vertices  $\{0_1, 1_1, 2_1, 3_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges between these two sets, and the edges between the vertices of the latter set will form a  $K_3 \vee 4K_1$ . By Lemma 1.3.6,  $K_3 \vee 4K_1$  has an  $S_5$ -decomposition,  $S'$ . Let  $S'$  be formed by the stars  $(0_2; 1_2, 0_1, 1_1, 2_1, 3_1)$ ,  $(1_2; 2_2, 0_1, 1_1, 2_1, 3_1)$ , and  $(2_2; 0_2, 0_1, 1_1, 2_1, 3_1)$ . Now, the set of vertices  $\{4_1, 5_1, 6_1, 7_1, 8_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , and the edges between these two sets form a complete bipartite graph with one part of size a multiple of 5. Hence, by Lemma 1.3.5, this graph has an  $S_5$ -decomposition,  $T'$ . Let  $T'$  be formed by the stars  $(0_2; 4_1, 5_1, 6_1, 7_1, 8_1)$ ,  $(1_2; 4_1, 5_1, 6_1, 7_1, 8_1)$ , and  $(2_2; 4_1, 5_1, 6_1, 7_1, 8_1)$ . Therefore,  $R' \cup S' \cup T'$  forms a maximum packing for  $K_{13}$  with the 3-star  $(9_1; 0_2, 1_2, 2_2)$  as the leave graph.

Substituting the star  $(0_2; 4_1, 5_1, 6_1, 7_1, 9_1)$  for  $(0_2; 4_1, 5_1, 6_1, 7_1, 8_1)$  in the packing  $R' \cup S' \cup T'$  gives us a maximum packing  $U$  of  $K_{13}$  with  $P_3 + K_2$  as the leave graph. The edges of the leave graph are  $\{0_2, 8_1\}$ ,  $\{9_1, 1_2\}$ , and  $\{9_1, 2_2\}$ .

Substituting the star  $(1_2; 2_2, 0_1, 1_1, 2_1, 9_1)$  for  $(1_2; 2_2, 0_1, 1_1, 2_1, 3_1)$  in the packing  $U$  results in a maximum packing for  $K_{13}$  with  $3K_2$  as the leave graph. The edges of the leave graph are  $\{0_2, 8_1\}$ ,  $\{3_1, 1_2\}$ , and  $\{9_1, 2_2\}$ .

Finally, considering the packing  $R' \cup S' \cup T'$  and substituting the star  $(0_2; 0_1, 1_1, 2_1, 3_1, 9_1)$  for  $(0_2; 1_2, 0_1, 1_1, 2_1, 3_1)$  gives us a maximum packing for  $K_{13}$  with the leave graph  $P_4$ . In fact, the leave graph is the path  $(0_2, 1_2, 9_1, 2_2)$ . This completes the proof in this case. ■

### 2.3.2 The Spectrum of Excess Graphs for 5-stars

In Section 2.2.1, we showed how to achieve all possible leave graphs in packing a complete graph with 5-stars. Now, we prove that all possible excess graphs in covering the complete graph with 5-stars are also achievable. Refer to Table 2.5 at the end of this section for all possible leave and excess graphs in different congruence classes.

**Theorem 2.3.2** *Let  $n \geq 10$  be an integer and let the excess graph in a minimum  $S_5$ -covering of the complete graph  $K_n$  have  $i$  edges. For any graph  $H$  with  $i$  edges there exists a minimum  $S_5$ -covering of  $K_n$  such that the excess graph is isomorphic to  $H$ , except for the excess graph  $K_2^4$  which is not achievable for  $n = 12$ .*

**Proof.** The complete graph  $K_n$  has an  $S_5$ -decomposition for  $n \equiv 0, 1, 5$ , or  $6 \pmod{10}$  by Theorem 1.3.1. We show that for the remaining cases we have minimum coverings with all the possible excess graphs.

By Lemma 1.3.10, we only need to prove the theorem for the cases when  $n \equiv 2, 3$ , and  $4 \pmod{10}$ . Also by Lemma 1.3.10, it suffices to consider the cases  $n = 12$ ,  $n = 13$ , and  $n = 14$ . However, for the excess graph  $K_2^4$  we need to consider  $n = 17$  as well.

Case 1.  $n = 12$ . By Theorem 1.3.3, the excess graph has 4 edges in this case. Figure 2.6 shows all possible excess graphs with 4 edges ( $E_i$  demonstrates the  $i$ th excess graph). Let  $P$  be a maximum  $S_5$ -packing of  $K_n$ . Since the leave graph in a

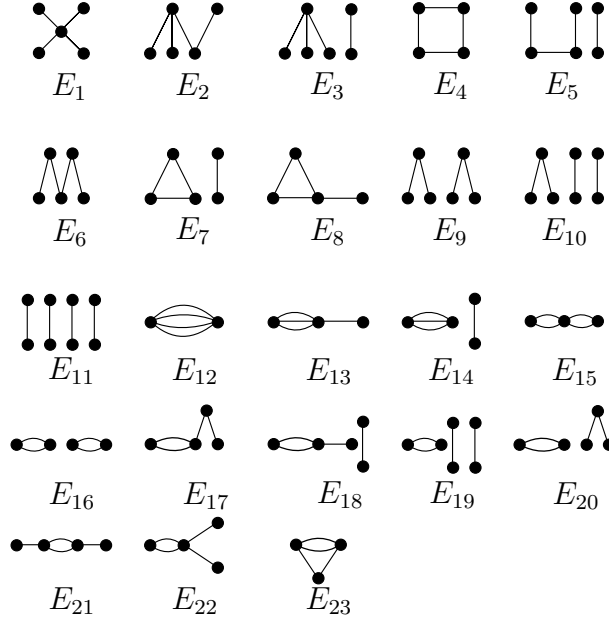


Figure 2.6: All possible 4-edge excess graphs

maximum  $S_5$ -packing is a single edge by Theorem 1.3.2, if we add a 5-star including that single edge, we obtain  $E_1$  as the excess graph.

In order to achieve  $E_{14}$ , write  $K_{12} = K_9 \vee K_3$ . Label the vertices of  $K_9 \vee K_3$  with a  $(\mathbb{Z}_9, \mathbb{Z}_3)$ -labeling. The following stars form a maximum  $S_5$ -packing,  $R$ , for  $K_9$  with

the single edge  $\{7_1, 8_1\}$  as the leave graph.

$$\begin{aligned} & (0_1; 1_1, 2_1, 3_1, 7_1, 8_1), (1_1; 2_1, 3_1, 4_1, 7_1, 8_1), (2_1; 3_1, 4_1, 5_1, 7_1, 8_1), \\ & (3_1; 4_1, 5_1, 6_1, 7_1, 8_1), (4_1; 5_1, 6_1, 0_1, 7_1, 8_1), (5_1; 6_1, 0_1, 1_1, 7_1, 8_1), \\ & (6_1; 0_1, 1_1, 2_1, 7_1, 8_1) \end{aligned}$$

Moreover, the following stars form a minimum  $S_5$ -covering,  $S$ , of the remaining edges with the edges  $\{0_2, 1_2\}$  used three times and the edge  $\{6_1, 8_1\}$  as the excess graph, which forms a graph isomorphic to  $E_{14}$ .

$$\begin{aligned} & (0_2; 1_2, 0_1, 1_1, 2_1, 3_1), (1_2; 0_2, 0_1, 1_1, 2_1, 3_1), (0_2; 1_2, 4_1, 5_1, 6_1, 7_1), \\ & (1_2; 0_2, 4_1, 5_1, 6_1, 7_1), (2_2; 0_2, 0_1, 1_1, 2_1, 3_1), (2_2; 1_2, 4_1, 5_1, 6_1, 7_1), \\ & (8_1; 6_1, 7_1, 0_2, 1_2, 2_2) \end{aligned}$$

Therefore,  $R \cup S$  forms a minimum  $S_5$ -covering for  $K_{12}$  with the excess graph  $E_{14}$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used three times and the edge  $\{6_1, 8_1\}$ . Figure 2.7 illustrates the last step in achieving a minimum  $S_5$ -covering of  $K_{12}$  with the excess graph  $E_{14}$ . Each thick line connected to an oval takes the place of a 4-star.

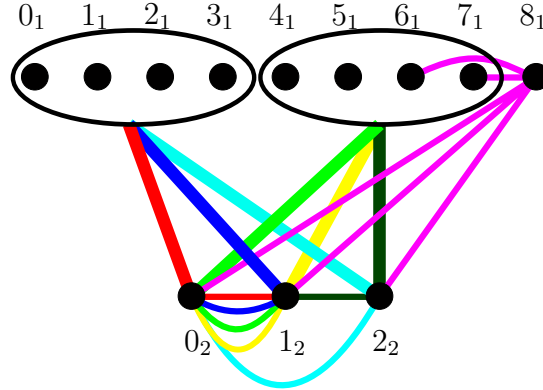


Figure 2.7: The last step in achieving a minimum  $S_5$ -covering of  $K_{12}$  with the excess graph  $E_{14}$

Substituting the stars  $(0_1; 1_1, 2_1, 3_1, 7_1, 0_2)$  and  $(8_1; 0_1, 7_1, 0_2, 1_2, 2_2)$  for  $(0_1; 1_1, 2_1, 3_1, 7_1, 8_1)$  and  $(8_1; 6_1, 7_1, 0_2, 1_2, 2_2)$  respectively, leads to a minimum  $S_5$ -covering,  $U$ , with the excess graph  $E_{13}$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used three

times and the edge  $\{0_1, 0_2\}$ .

Now, we obtain all possible excess graphs by substitution of some stars with some other ones in the coverings  $R \cup S$  and  $U$ . The substitutions are given in Tables 2.2 and 2.3. In fact, the excess graphs  $E_2, E_3, E_4, E_5, E_6, E_7, E_{10}, E_{11}, E_{17}, E_{18}, E_{19}$ , and  $E_{20}$  are achieved from the covering  $R \cup S$ , and the excess graphs  $E_8, E_9, E_{15}, E_{16}, E_{21}, E_{22}$ , and  $E_{23}$  are achieved from the covering  $U$ .

New star(s)	Previous star(s)	Edges of the excess graph	Excess
$(0_2; 6_1, 0_1, 1_1, 2_1, 3_1)$ $(0_2; 0_1, 4_1, 5_1, 6_1, 7_1)$	$(0_2; 1_2, 0_1, 1_1, 2_1, 3_1)$ $(0_2; 1_2, 4_1, 5_1, 6_1, 7_1)$	$\{6_1, 8_1\}, \{0_1, 0_2\}, \{6_1, 0_2\}, \{0_2, 1_2\}$	$E_2$
$(0_2; 4_1, 0_1, 1_1, 2_1, 3_1)$ $(0_2; 0_1, 4_1, 5_1, 6_1, 7_1)$	$(0_2; 1_2, 0_1, 1_1, 2_1, 3_1)$ $(0_2; 1_2, 4_1, 5_1, 6_1, 7_1)$	$\{6_1, 8_1\}, \{0_1, 0_2\}, \{4_1, 0_2\}, \{0_2, 1_2\}$	$E_3$
$(0_2; 6_1, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 8_1, 0_1, 1_1, 2_1, 3_1)$	$(0_2; 1_2, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 0_2, 0_1, 1_1, 2_1, 3_1)$	$\{6_1, 8_1\}, \{6_1, 0_2\}, \{8_1, 1_2\}, \{0_2, 1_2\}$	$E_4$
$(0_2; 4_1, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 5_1, 0_1, 1_1, 2_1, 3_1)$	$(0_2; 1_2, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 0_2, 0_1, 1_1, 2_1, 3_1)$	$\{6_1, 8_1\}, \{4_1, 0_2\}, \{5_1, 1_2\}, \{0_2, 1_2\}$	$E_5$
$(0_2; 6_1, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 5_1, 0_1, 1_1, 2_1, 3_1)$	$(0_2; 1_2, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 0_2, 0_1, 1_1, 2_1, 3_1)$	$\{6_1, 8_1\}, \{5_1, 1_2\}, \{6_1, 0_2\}, \{0_2, 1_2\}$	$E_6$
$(1_2; 2_2, 0_1, 1_1, 2_1, 3_1)$ $(0_2; 2_2, 4_1, 5_1, 6_1, 7_1)$	$(1_2; 0_2, 0_1, 1_1, 2_1, 3_1)$ $(0_2; 1_2, 4_1, 5_1, 6_1, 7_1)$	$\{6_1, 8_1\}, \{0_2, 1_2\}, \{0_2, 2_2\}, \{1_2, 2_2\}$	$E_7$
$(0_2; 4_1, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 5_1, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 2_2, 4_1, 5_1, 6_1, 7_1)$	$(0_2; 1_2, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 0_2, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 0_2, 4_1, 5_1, 6_1, 7_1)$	$\{6_1, 8_1\}, \{4_1, 0_2\}, \{5_1, 1_2\}, \{1_2, 2_2\}$	$E_{10}$
$(0_2; 4_1, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 5_1, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 2_2, 4_1, 5_1, 6_1, 7_1)$ $(2_2; 3_1, 4_1, 5_1, 6_1, 7_1)$	$(0_2; 1_2, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 0_2, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 0_2, 4_1, 5_1, 6_1, 7_1)$ $(2_2; 1_2, 4_1, 5_1, 6_1, 7_1)$	$\{6_1, 8_1\}, \{3_1, 2_2\}, \{4_1, 0_2\}, \{5_1, 1_2\}$	$E_{11}$
$(0_2; 8_1, 4_1, 5_1, 6_1, 7_1)$	$(0_2; 1_2, 4_1, 5_1, 6_1, 7_1)$	$\{6_1, 8_1\}, \{8_1, 0_2\}, \{0_2, 1_2\}$ (twice)	$E_{17}$
$(0_2; 2_1, 4_1, 5_1, 6_1, 7_1)$	$(0_2; 1_2, 4_1, 5_1, 6_1, 7_1)$	$\{6_1, 8_1\}, \{0_2, 2_1\}, \{0_2, 1_2\}$ (twice)	$E_{18}$
$(1_2; 2_2, 4_1, 5_1, 6_1, 7_1)$ $(2_2; 3_1, 4_1, 5_1, 6_1, 7_1)$	$(1_2; 0_2, 4_1, 5_1, 6_1, 7_1)$ $(2_2; 1_2, 4_1, 5_1, 6_1, 7_1)$	$\{6_1, 8_1\}, \{3_1, 2_2\}, \{0_2, 1_2\}$ (twice)	$E_{19}$
$(0_2; 2_2, 0_1, 1_1, 2_1, 3_1)$ $(2_2; 6_1, 0_1, 1_1, 2_1, 3_1)$	$(0_2; 1_2, 0_1, 1_1, 2_1, 3_1)$ $(2_2; 0_2, 0_1, 1_1, 2_1, 3_1)$	$\{6_1, 8_1\}, \{6_1, 2_2\}, \{0_2, 1_2\}$ (twice)	$E_{20}$

Table 2.2: Substitutions in the  $S_5$ -covering  $R \cup S$  to obtain different excess graphs for  $n = 12$

Now, we prove that for  $n = 12$ , the excess graph  $E_{12} = K_2^4$  is not achievable. Assume to the contrary that  $Q$  is a minimum  $S_5$ -covering of  $K_{12}$  with the excess graph  $E_{12}$ . Let  $x$  and  $y$  be the end vertices of the four multiple edges of the excess graph. Since the four multiple edges form the excess graph,  $Q$  contains five multiple edges  $\{x, y\}$ . We claim that each of the vertices  $x$  and  $y$  can be at most the center of two stars of  $Q$  containing the edge  $\{x, y\}$ . Assume to the contrary that  $x$  is the center of three such stars. Hence, there exist at least three disjoint sets of four vertices

New star(s)	Previous star(s)	Edges of the excess graph	Excess
$(1_2; 2_2, 0_1, 1_1, 2_1, 3_1)$ $(0_2; 2_2, 4_1, 5_1, 6_1, 7_1)$	$(1_2; 0_2, 0_1, 1_1, 2_1, 3_1)$ $(0_2; 1_2, 4_1, 5_1, 6_1, 7_1)$	$\{0_1, 0_2\}, \{0_2, 1_2\}, \{0_2, 2_2\}, \{1_2, 2_2\}$	$E_8$
$(0_2; 4_1, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 2_1, 4_1, 5_1, 6_1, 7_1)$ $(1_2; 5_1, 0_1, 1_1, 2_1, 3_1)$	$(0_2; 1_2, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 0_2, 4_1, 5_1, 6_1, 7_1)$ $(1_2; 0_2, 0_1, 1_1, 2_1, 3_1)$	$\{0_1, 0_2\}, \{4_1, 0_2\}, \{2_1, 1_2\}, \{5_1, 1_2\}$	$E_9$
$(0_2; 0_1, 4_1, 5_1, 6_1, 7_1)$	$(0_2; 1_2, 4_1, 5_1, 6_1, 7_1)$	$\{0_1, 0_2\}(\text{twice}), \{0_2, 1_2\}(\text{twice})$	$E_{15}$
$(0_2; 0_1, 4_1, 5_1, 6_1, 7_1)$ $(1_2; 8_1, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 8_1, 4_1, 5_1, 6_1, 7_1)$	$(0_2; 1_2, 4_1, 5_1, 6_1, 7_1)$ $(1_2; 0_2, 0_1, 1_1, 2_1, 3_1)$ $(1_2; 0_2, 4_1, 5_1, 6_1, 7_1)$	$\{8_1, 1_2\}(\text{twice}), \{0_1, 0_2\}(\text{twice})$	$E_{16}$
$(1_2; 4_1, 0_1, 1_1, 2_1, 3_1)$	$(1_2; 0_2, 0_1, 1_1, 2_1, 3_1)$	$\{0_1, 0_2\}, \{4_1, 1_2\}, \{0_2, 1_2\}(\text{twice})$	$E_{21}$
$(0_2; 1_1, 4_1, 5_1, 6_1, 7_1)$	$(0_2; 1_2, 4_1, 5_1, 6_1, 7_1)$	$\{0_1, 0_2\}, \{1_1, 0_2\}, \{0_2, 1_2\}(\text{twice})$	$E_{22}$
$(1_2; 0_1, 4_1, 5_1, 6_1, 7_1)$	$(1_2; 0_2, 4_1, 5_1, 6_1, 7_1)$	$\{0_1, 0_2\}, \{0_1, 1_2\}, \{0_2, 1_2\}(\text{twice})$	$E_{23}$

Table 2.3: Substitutions in the  $S_5$ -covering  $U$  to obtain different excess graphs for  $n = 12$

other than  $x$  and  $y$ . This contradicts  $n = 12$  and hence, our claim is true. Therefore, there are at most four multiple edges between  $x$  and  $y$  in  $Q$ , which shows there is no minimum  $S_5$ -covering of  $K_{12}$  with the excess graph  $E_{12}$ .

Case 2.  $n = 13$ .

In this case, the excess graph has two edges by Theorem 1.3.3. Hence, the possible excess graphs are  $P_3$ ,  $2K_2$ , and  $K_2^2$ .

The excess graph  $P_3$  is easily achievable by adding a 5-star to a maximum packing of  $K_{13}$  with the leave graph  $S_3$ .

In order to obtain the excess graph  $2K_2$ , write  $K_{13} = K_9 \vee K_4$ . Label the vertices of  $K_9 \vee K_4$  with a  $(\mathbb{Z}_9, \mathbb{Z}_4)$ -labeling. Let  $R$  be a maximum  $S_5$ -packing of  $K_9$  with the single edge  $\{7_1, 8_1\}$  as the leave graph. Consider the set of vertices  $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , and the edges between these two sets form a complete bipartite graph with one part of size a multiple of 5. Hence, by Lemma 1.3.5, this bipartite graph has an  $S_5$ -decomposition,  $S$ . Furthermore, the following stars form a minimum covering,  $T$ , for the remaining graph with the edges  $\{0_2, 2_2\}$  and  $\{1_2, 3_2\}$  as the excess graph.

$$(0_2; 1_2, 2_2, 5_1, 6_1, 7_1), (1_2; 2_2, 3_2, 5_1, 6_1, 7_1), (2_2; 3_2, 0_2, 5_1, 6_1, 7_1), \\ (3_2; 0_2, 1_2, 5_1, 6_1, 7_1), (8_1; 7_1, 0_2, 1_2, 2_2, 3_2)$$

Therefore,  $R \cup S \cup T$  forms a minimum  $S_5$ -covering of  $K_{13}$  with the excess graph  $2K_2$ . The edges of the excess graph are  $\{0_2, 2_2\}$  and  $\{1_2, 3_2\}$ . Figure 2.8 illustrates the last step in achieving a minimum  $S_5$ -covering of  $K_{13}$  with the excess graph  $2K_2$ .

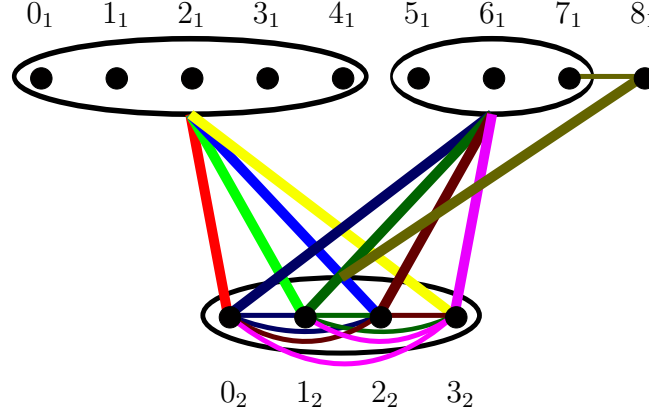


Figure 2.8: The last step in achieving a minimum  $S_5$ -covering of  $K_{13}$  with the excess graph  $2K_2$

In order to achieve the excess graph  $K_2^2$ , partition and label the vertices of  $K_{13}$  as before and let  $R$  be the same  $S_5$ -packing of  $K_9$  with the same edge as the leave graph. The following stars form a minimum covering,  $R''$ , of the remaining graph with the edges  $\{0_2, 1_2\}$  used twice, as the excess graph.

$$\begin{aligned} & (0_2; 1_2, 0_1, 1_1, 2_1, 3_1), (0_2; 1_2, 4_1, 5_1, 6_1, 7_1), (1_2; 0_2, 0_1, 1_1, 2_1, 3_1), \\ & (1_2; 2_2, 4_1, 5_1, 6_1, 7_1), (2_2; 0_2, 0_1, 1_1, 2_1, 3_1), (2_2; 3_2, 4_1, 5_1, 6_1, 7_1), \\ & (3_2; 0_2, 0_1, 1_1, 2_1, 3_1), (3_2; 1_2, 4_1, 5_1, 6_1, 7_1), (8_1; 7_1, 0_2, 1_2, 2_2, 3_2) \end{aligned}$$

Therefore,  $R \cup R''$  forms a minimum covering for  $K_{13}$  with the excess graph  $K_2^2$ . The edges of the excess graph are the multiple edges  $\{0_2, 1_2\}$ . Figure 2.9 illustrates the last step in achieving a minimum  $S_5$ -covering of  $K_{13}$  with the excess graph  $K_2^2$ .

Case 3.  $n = 14$

In this case, the excess graph has four edges by Theorem 1.3.3. Hence, the possible excess graphs are the ones shown in Figure 2.6. Again, since the leave graph in a maximum  $S_5$ -packing is a single edge, if we add a 5-star including that single edge, we obtain  $E_1$  as the excess graph.

In order to achieve the excess graph  $E_{12}$ , write  $K_{14} = K_9 \vee K_5$ . Label the vertices

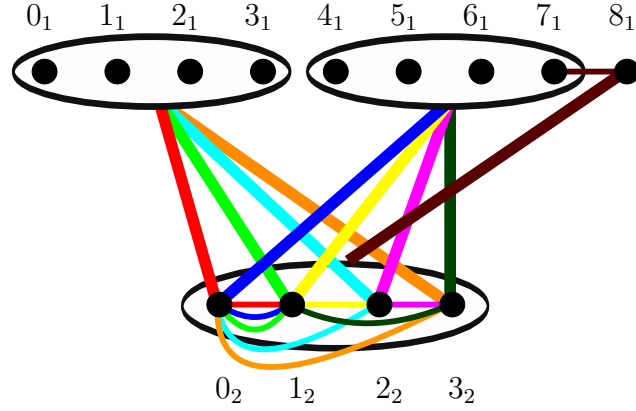


Figure 2.9: The last step in achieving a minimum  $S_5$ -covering of  $K_{13}$  with the excess graph  $K_2^2$

of  $K_9 \vee K_5$  with a  $(\mathbb{Z}_9, \mathbb{Z}_5)$ -labeling. Let  $R$  be a maximum  $S_5$ -packing of  $K_9$  and let the leave graph be the edge  $\{7_1, 8_1\}$ . The following stars form a minimum covering,  $S$ , of the remaining edges with the edges  $\{0_2, 1_2\}$  used four times, as the excess graph.

$$\begin{aligned}
 & (0_2; 1_2, 0_1, 1_1, 2_1, 3_1), (0_2; 1_2, 4_1, 5_1, 6_1, 7_1), (1_2; 0_2, 0_1, 1_1, 2_1, 3_1), \\
 & (1_2; 0_2, 4_1, 5_1, 6_1, 7_1), (0_2; 1_2, 2_2, 3_2, 4_2, 8_1), (2_2; 1_2, 0_1, 1_1, 2_1, 3_1), \\
 & (2_2; 3_2, 4_1, 5_1, 6_1, 7_1), (3_2; 4_2, 0_1, 1_1, 2_1, 3_1), (3_2; 1_2, 4_1, 5_1, 6_1, 7_1), \\
 & (4_2; 1_2, 0_1, 1_1, 2_1, 3_1), (4_2; 2_2, 4_1, 5_1, 6_1, 7_1), (8_1; 7_1, 1_2, 2_2, 3_2, 4_2)
 \end{aligned}$$

Therefore,  $R \cup S$  forms a minimum  $S_5$ -covering of  $K_{14}$  with the excess graph  $E_{12}$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used four times. Figure 2.10 illustrates the last step in achieving a minimum  $S_5$ -covering of  $K_{14}$  with the excess graph  $E_{12}$ .

Consider the covering  $R \cup S$ . Table 2.4 shows the star substitutions in  $R \cup S$  needed to achieve each excess graph except for  $E_{11}$ .

In order to achieve the excess graph  $E_{11}$ , write  $K_{14} = K_{10} \vee K_4$ . Label the vertices of  $K_{10} \vee K_4$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_4)$ -labeling. The complete graph  $K_{10}$  has an  $S_5$ -decomposition,  $U$ . The set of vertices  $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , and the edges between these sets form a complete bipartite graph with one part of size a multiple of 5. Hence, by Lemma 1.3.5, this bipartite graph has an  $S_5$ -decomposition,

New star(s)	Previous star(s)	Edges of the excess graph	Excess
(0 <sub>2</sub> ; 0 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 2 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	{0 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }, {1 <sub>2</sub> , 2 <sub>2</sub> }	$E_2$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 2 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (2 <sub>2</sub> ; 5 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (2 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	{4 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>1</sub> , 0 <sub>2</sub> }, {5 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_3$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	{0 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>1</sub> , 1 <sub>2</sub> }, {4 <sub>1</sub> , 1 <sub>2</sub> }	$E_4$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 2 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (2 <sub>2</sub> ; 5 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (2 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	{0 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 1 <sub>2</sub> }, {5 <sub>1</sub> , 2 <sub>2</sub> }	$E_5$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 8 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	{0 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 1 <sub>2</sub> }, {8 <sub>1</sub> , 1 <sub>2</sub> }	$E_6$
(0 <sub>2</sub> ; 2 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>2</sub> ; 2 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 3 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (3 <sub>2</sub> ; 4 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (3 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	{0 <sub>2</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 2 <sub>2</sub> }, {1 <sub>2</sub> , 2 <sub>2</sub> }, {3 <sub>2</sub> , 4 <sub>2</sub> }	$E_7$
(0 <sub>2</sub> ; 2 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>2</sub> ; 2 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 3 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	{0 <sub>2</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 2 <sub>2</sub> }, {1 <sub>2</sub> , 2 <sub>2</sub> }, {1 <sub>2</sub> , 3 <sub>2</sub> }	$E_8$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 1 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	{0 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 0 <sub>2</sub> }, {1 <sub>1</sub> , 1 <sub>2</sub> }, {5 <sub>1</sub> , 1 <sub>2</sub> }	$E_9$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 2 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (2 <sub>2</sub> ; 6 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (2 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	{0 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 0 <sub>2</sub> }, {5 <sub>1</sub> , 1 <sub>2</sub> }, {6 <sub>1</sub> , 2 <sub>2</sub> }	$E_{10}$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	{0 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (3 times)	$E_{13}$
(1 <sub>2</sub> ; 2 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (2 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (2 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	{0 <sub>2</sub> , 1 <sub>2</sub> } (3 times), {4 <sub>1</sub> , 2 <sub>2</sub> }	$E_{14}$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 0 <sub>1</sub> , 8 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 8 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> )	{0 <sub>1</sub> , 0 <sub>2</sub> } (twice), {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{15}$
(1 <sub>2</sub> ; 2 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (2 <sub>2</sub> ; 3 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 3 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (3 <sub>2</sub> ; 2 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (2 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (3 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	{0 <sub>2</sub> , 1 <sub>2</sub> } (twice), {2 <sub>2</sub> , 3 <sub>2</sub> } (twice)	$E_{16}$
(1 <sub>2</sub> ; 2 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>2</sub> ; 3 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (3 <sub>2</sub> ; 2 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (3 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	{0 <sub>2</sub> , 1 <sub>2</sub> } (twice), {1 <sub>2</sub> , 2 <sub>2</sub> }, {2 <sub>2</sub> , 3 <sub>2</sub> }	$E_{17}$
(1 <sub>2</sub> ; 2 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>2</sub> ; 4 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (4 <sub>2</sub> ; 3 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (4 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	{0 <sub>2</sub> , 1 <sub>2</sub> } (twice), {1 <sub>2</sub> , 2 <sub>2</sub> }, {3 <sub>2</sub> , 4 <sub>2</sub> }	$E_{18}$



New star(s)	Previous star(s)	Edges of the excess graph	Excess
(1 <sub>2</sub> ; 2 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (2 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 4 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (4 <sub>2</sub> ; 3 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (2 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (4 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	{4 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }(twice), {3 <sub>2</sub> , 4 <sub>2</sub> }	$E_{19}$
(1 <sub>2</sub> ; 3 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (3 <sub>2</sub> ; 2 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>2</sub> ; 4 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (4 <sub>2</sub> ; 3 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (3 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> ) (4 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	{0 <sub>2</sub> , 1 <sub>2</sub> }(twice), {2 <sub>2</sub> , 3 <sub>2</sub> }, {3 <sub>2</sub> , 4 <sub>2</sub> }	$E_{20}$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 3 <sub>1</sub> )	{0 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }(twice)	$E_{21}$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>1</sub> , 8 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 8 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> )	{0 <sub>1</sub> , 0 <sub>2</sub> }, {1 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }(twice)	$E_{22}$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> )	{0 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }(twice)	$E_{23}$

Table 2.4: Substitutions in the  $S_5$ -covering  $R \cup S$  to obtain different excess graphs for  $n = 14$

V. Moreover, the following stars form a minimum covering,  $W$ , for the remaining graph with the edges  $\{8_1, 0_1\}$ ,  $\{9_1, 1_1\}$ ,  $\{0_2, 2_2\}$ , and  $\{1_2, 3_2\}$  as the excess graph.

$$(0_2; 1_2, 2_2, 5_1, 6_1, 7_1), (1_2; 2_2, 3_2, 5_1, 6_1, 7_1), (2_2; 3_2, 0_2, 5_1, 6_1, 7_1), \\ (3_2; 0_2, 1_2, 5_1, 6_1, 7_1), (8_1; 0_1, 0_2, 1_2, 2_2, 3_2), (9_1; 1_1, 0_2, 1_2, 2_2, 3_2)$$

Therefore,  $U \cup V \cup W$  is a minimum  $S_5$ -covering of  $K_{14}$  with the excess graph  $E_{11}$ . Figure 2.11 illustrates the last step in achieving a minimum  $S_5$ -covering of  $K_{14}$  with the excess graph  $E_{11}$ .

In order to achieve the excess graph  $E_{12}$  for  $n \geq 17$  where  $n \equiv 2 \pmod{5}$ , it suffices to achieve this excess graph for  $n = 17$  by Lemma 1.3.10. Let  $n = 17$ . Write  $K_{17} = K_{14} \vee K_3$  and label the vertices of  $K_{14} \vee K_3$  with a  $(\mathbb{Z}_{14}, \mathbb{Z}_3)$ -labeling. As shown in Case 3, the graph  $K_{14}$  has an  $S_5$ -covering,  $R$ , with the excess graph  $E_{12}$ . The set of vertices  $\{0_1, 1_1, \dots, 9_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , and the edges between these two sets form a complete bipartite graph with one part of size a multiple of 5. Hence, by Lemma 1.3.5, this complete bipartite graph has an  $S_5$ -decomposition,  $S$ . Now, the set of vertices  $\{10_1, 11_1, 12_1, 13_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges between these two sets, and the edges within the second set form a graph  $K_3 \vee 4K_1$ , which has an  $S_5$ -decomposition,  $T$ , by Lemma 1.3.6. Therefore,  $R \cup S \cup T$  forms a minimum  $S_5$ -covering of  $K_{17}$  with the excess graph  $E_{12}$ . ■

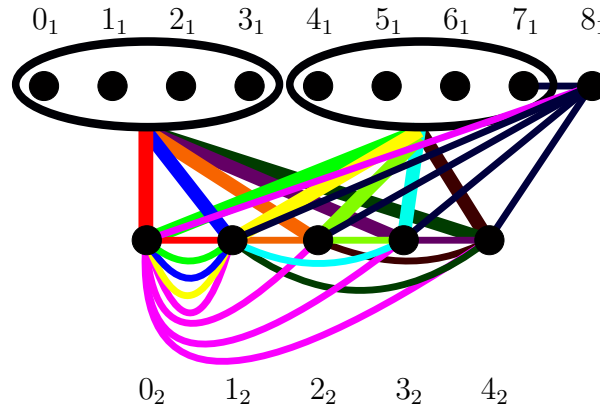


Figure 2.10: The last step in achieving a minimum  $S_5$ -covering of  $K_{14}$  with the excess graph  $E_{12}$

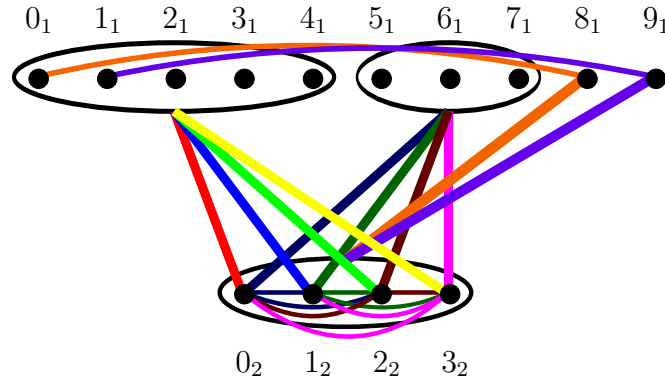


Figure 2.11: The last step in achieving a minimum  $S_5$ -covering of  $K_{14}$  with the excess graph  $E_{11}$

$n \pmod{10}$	Possible leave graphs	Possible excess graphs
0	$\emptyset$	$\emptyset$
1	$\emptyset$	$\emptyset$
2	$K_2$	the graphs in Figure 2.6 except for $E_{12}$ when $n = 12$
3	$S_3, K_3, P_4, 3K_2$ , and $P_3 + K_2$	$P_3, 2K_2$ , and $K_2^2$
4	$K_2$	the graphs in Figure 2.6
5	$\emptyset$	$\emptyset$
6	$\emptyset$	$\emptyset$
7	$K_2$	the graphs in Figure 2.6
8	$S_3, K_3, P_4, 3K_2$ , and $P_3 + K_2$	$P_3, 2K_2$ , and $K_2^2$
9	$K_2$	the graphs in Figure 2.6

Table 2.5: The spectrum of the leave graphs (for  $n \geq 9$ ) and excess graphs (for  $n \geq 10$ ) for 5-stars

Table 2.5 illustrates the spectrum of the leave graphs (for  $n \geq 9$ ) and excess graphs (for  $n \geq 10$ ) for 5-stars.

## Chapter 3

# The Spectrum of Leave Graphs for Trees with up to Five Edges

In this chapter, we will find all possible leave graphs in packings of the complete graph with trees that have up to five edges. If a tree has one edge, then the tree is a single edge, and any complete graph can be decomposed into single edges. The only tree with two edges is  $P_3$ . The leave graph in a maximum  $P_3$ -packing of any complete graph has at most one edge [37], in which case, the only possible leave graph is  $K_2$ . Also any tree  $T$  with three edges has four vertices, and the leave graph in any maximum  $T$ -packing of any complete graph has at most one edge [37], and the only possible leave graph will be  $K_2$ .

With the above explanation, we only need to consider trees with four and five edges.

### 3.1 The Spectrum of Leave Graphs for Trees with Four Edges

**Theorem 3.1.1** *Let  $n \geq 7$  be an integer,  $T$  be any tree with four edges, and let the leave graph in a maximum  $T$ -packing of the complete graph  $K_n$  have  $i$  edges. For any graph  $H$  with  $i$  edges there exists a maximum  $T$ -packing of  $K_n$  such that the leave graph is isomorphic to  $H$ .*

**Proof.** Let  $n \geq 7$  be an integer and  $T$  be any tree with four edges. For  $n \equiv 0, 1 \pmod{8}$ , the complete graph  $K_n$  has a  $T$ -decomposition by Theorem 1.3.1. For  $n \equiv 2, 7 \pmod{8}$ , the leave graph is a single edge by Theorem 1.3.2. We show that for  $n \equiv 3, 4, 5, 6 \pmod{8}$ , we can achieve every possible leave graph. All trees with four edges are demonstrated in Figure 3.1, and  $A$  is denoted by  $(x_1; x_2, x_3, x_4 - x_5)$ .

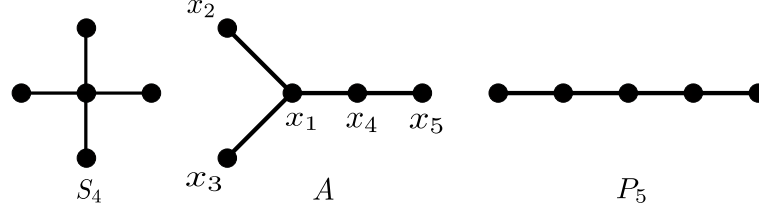


Figure 3.1: All trees with four edges

The theorem is proved for  $T = S_4$  (see Theorem 2.2.1). We need to prove the result for  $A$  and  $P_5$ . For both cases, we prove the theorem considering congruency classes modulo 8.

Case 1.  $n \equiv 3 \pmod{8}$ ,  $T = A$

The leave graph has three edges by Theorem 1.3.2. Therefore, the possible leave graphs are  $K_3$ ,  $S_3$ ,  $P_4$ ,  $3K_2$ , and  $P_3 + K_2$ . By Corollary 1.3.15, it suffices to achieve all possible leave graphs for  $K_{11}$ . In fact, for  $n = 8k + 3$  where  $k \geq 1$  is an integer, we write  $K_n = K_{8(k-1)} \vee K_{11}$ . Let  $R$  be an  $A$ -decomposition of  $K_{8(k-1)}$  and  $S$  be a maximum  $A$ -packing of  $K_{11}$  with the leave graph  $H$  where  $H$  is any of the possible leave graphs. By Lemma 1.3.14, the graph  $K_{8(k-1), 11}$  has an  $A$ -decomposition,  $U$ . Therefore,  $R \cup S \cup U$  is a maximum  $A$ -packing of  $K_n$  with the leave graph  $H$ .

The leave graph  $P_4$  was obtained by Roditty [39]. In order to obtain the leave graph  $K_3$ , write  $K_{11} = K_8 \vee K_3$ . Label the vertices of  $K_8 \vee K_3$  with a  $(\mathbb{Z}_8, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.1,  $K_8$  has an  $A$ -decomposition,  $R$ . By Lemma 1.3.14,  $K_{8,3}$  has an  $A$ -decomposition,  $S$ . Let  $S$  be formed by the following trees.

$$(0_2; 0_1, 1_1, 2_1 - 1_2), (1_2; 0_1, 3_1, 1_1 - 2_2), (2_2; 0_1, 2_1, 3_1 - 0_2), \\ (0_2; 4_1, 5_1, 6_1 - 1_2), (1_2; 4_1, 7_1, 5_1 - 2_2), (2_2; 4_1, 6_1, 7_1 - 0_2)$$

Therefore,  $R \cup S$  forms a maximum  $A$ -packing of  $K_{11}$  with the leave graph  $K_3$ . The edges of the leave graph are  $\{0_2, 1_2\}$ ,  $\{0_2, 2_2\}$ , and  $\{1_2, 2_2\}$ .

In order to obtain the leave graphs  $P_3 + K_2$  and  $3K_2$ , we replace some of the trees in this packing with others. Table 3.1 shows the required substitutions.

New tree(s)	Previous tree(s)	Edges of the leave graph	Leave
$(0_2; 2_2, 1_1, 2_1 - 1_2)$ $(1_2; 0_1, 0_2, 1_1 - 2_2)$	$(0_2; 0_1, 1_1, 2_1 - 1_2)$ $(1_2; 0_1, 3_1, 1_1 - 2_2)$	$\{0_1, 0_2\}, \{1_2, 3_1\}, \{1_2, 2_2\}$	$P_3 + K_2$
$(0_2; 2_2, 1_1, 2_1 - 1_2),$ $(1_2; 0_1, 0_2, 1_1 - 2_2),$ $(2_2; 0_1, 1_2, 3_1 - 0_2)$	$(0_2; 0_1, 1_1, 2_1 - 1_2),$ $(1_2; 0_1, 3_1, 1_1 - 2_2),$ $(2_2; 0_1, 2_1, 3_1 - 0_2)$	$\{0_1, 0_2\}, \{1_2, 3_1\}, \{2_1, 2_2\}$	$3K_2$

Table 3.1: Substitutions in the  $A$ -packing  $R \cup S$  to obtain different leave graphs for  $n = 11$

In order to obtain  $S_3$  as the leave graph, label the vertices as before. Let  $R$  be the  $A$ -decomposition of  $K_8$ . The set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges between these two sets, and the edges within the second set, form a graph  $K_3 \vee 3K_1$ . The following trees form an  $A$ -decomposition,  $S$ , of the graph  $K_3 \vee 3K_1$ .

$$(0_2; 0_1, 1_1, 1_2 - 2_1), (1_2; 0_1, 1_1, 2_2 - 2_1), (2_2; 0_1, 1_1, 0_2 - 2_1)$$

The complete bipartite graph with one part of vertices  $3_1, 4_1, 5_1, 6_1$  and another part  $0_2, 1_2, 2_2$  has an  $A$ -decomposition,  $U$ , by Lemma 1.3.14. Therefore,  $R \cup S \cup U$  forms a maximum  $A$ -packing of  $K_{11}$  with the leave graph  $S_3$ . The edges of the leave graph are  $\{7_1, 0_2\}, \{7_1, 1_2\}$ , and  $\{7_1, 2_2\}$ .

Case 2.  $n \equiv 3 \pmod{8}$ ,  $T = P_5$

By Corollary 1.3.15, it suffices to achieve all possible leave graphs for  $n = 11$ . The leave graph has three edges by Theorem 1.3.2. Hence, the possible leave graphs are  $K_3$ ,  $S_3$ ,  $P_4$ ,  $P_3 + K_2$ , and  $3K_2$ . In order to obtain the leave graph  $K_3$ , write  $K_{11} = K_8 \vee K_3$ . Label the vertices of  $K_8 \vee K_3$  with a  $(\mathbb{Z}_8, \mathbb{Z}_3)$ -labeling. Let  $R$  be a  $P_5$ -decomposition of  $K_8$ . The graph  $K_{8,3}$  has a  $P_5$ -decomposition,  $S$ , by Lemma 1.3.14. Therefore,  $R \cup S$  forms a maximum  $P_5$ -packing with the leave graph  $K_3$ . The edges of the leave graph are  $\{0_2, 1_2\}, \{0_2, 2_2\}$ , and  $\{1_2, 2_2\}$ .

In order to obtain the leave graph  $S_3$ , partition and label the vertices as above and let  $R$  be a  $P_5$ -decomposition of  $K_8$ . Consider the complete bipartite graph with one partite set  $\{3_1, 4_1, 5_1, 6_1\}$  and the other partite set  $\{0_2, 1_2, 2_2\}$ . This graph has a  $P_5$ -decomposition,  $S'$ , by Lemma 1.3.14. The set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges between these two sets, and the edges within the

second set form a graph  $K_3 \vee 3K_1$ . The following paths form a  $P_5$ -decomposition,  $U$ , for  $K_3 \vee 3K_1$ .

$$(1_2, 0_2, 0_1, 2_2, 1_1), (2_2, 1_2, 1_1, 0_2, 2_1), (0_2, 2_2, 2_1, 1_2, 0_1)$$

Therefore,  $R \cup S' \cup U$  forms a maximum  $P_5$ -packing of  $K_{11}$  with the leave graph  $S_3$ . In fact, the leave graph is the 3-star  $(7_1; 0_2, 1_2, 2_2)$ .

The other leave graphs can be achieved by substituting some paths with other ones in the packing  $R \cup S' \cup U$ . (See Table 3.2.)

New path(s)	Previous path(s)	Edges of the leave graph	Leave
$(7_1, 2_2, 2_1, 1_2, 0_1)$	$(0_2, 2_2, 2_1, 1_2, 0_1)$	$\{7_1, 0_2\}, \{7_1, 1_2\}, \{0_2, 2_2\}$	$P_4$
$(0_2, 2_2, 7_1, 1_2, 0_1)$	$(0_2, 2_2, 2_1, 1_2, 0_1)$	$\{7_1, 0_2\}, \{2_1, 2_2\}, \{1_2, 2_1\}$	$P_3 + K_2$
$(7_1, 0_2, 1_1, 1_2, 2_2)$ $(0_2, 2_2, 2_1, 1_2, 7_1)$	$(2_1, 0_2, 1_1, 1_2, 2_2)$ $(0_2, 2_2, 2_1, 1_2, 0_1)$	$\{0_1, 1_2\}, \{7_1, 2_2\}, \{0_2, 2_1\}$	$3K_2$

Table 3.2: Substitutions in the  $P_5$ -packing  $R \cup S' \cup U$  to obtain different leave graphs for  $n = 11$

Case 3.  $n \equiv 4 \pmod{8}$ ,  $T = A$

By Theorem 1.3.2, the leave graph has 2 edges in this case. So, the only possible leave graphs are  $P_3$  and  $2K_2$ . By Corollary 1.3.15, it suffices to achieve all possible leave graphs for  $K_{12}$ . Roditty showed how to obtain the leave graph  $P_3$  [39].

In order to achieve leave graph  $2K_2$ , write  $K_{12} = K_8 \vee K_4$ . Label the vertices of  $K_8 \vee K_4$  with a  $(\mathbb{Z}_8, \mathbb{Z}_4)$ -labeling. By Theorem 1.3.1,  $K_8$  has an  $A$ -decomposition,  $R$ . Consider the set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , the edges between these two sets, and the edges within the latter. These vertices and edges form a graph  $K_4 \vee 3K_1$ . The following trees construct a maximum  $A$ -packing,  $S$ , of the graph  $K_4 \vee 3K_1$  with the edges  $\{0_2, 2_2\}$  and  $\{1_2, 3_2\}$ , which make a graph  $2K_2$ , as the leave graph. (See Figure 3.2.)

$$(0_2; 1_2, 0_1, 1_1 - 2_2), (1_2; 2_2, 0_1, 1_1 - 3_2), (2_2; 3_2, 0_1, 2_1 - 0_2), (3_2; 0_2, 0_1, 2_1 - 1_2)$$

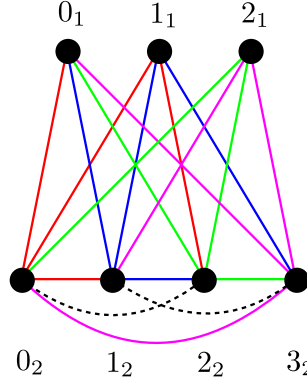


Figure 3.2: An  $A$ -packing of  $K_4 \vee 3K_1$  with the leave graph  $2K_2$

Now, consider the complete bipartite graph with partite sets  $\{3_1, 4_1, 5_1, 6_1, 7_1\}$  and  $\{0_2, 1_2, 2_2, 3_2\}$ . By Lemma 1.3.14, this bipartite graph has an  $A$ -decomposition,  $U$ . Therefore,  $R \cup S \cup U$  forms a maximum  $A$ -packing of  $K_{12}$  with the leave graph  $2K_2$ . The edges of the leave graph are  $\{0_2, 2_2\}$  and  $\{1_2, 3_2\}$ .

Case 4.  $n \equiv 4 \pmod{8}$ ,  $T = P_5$

By Corollary 1.3.15, it suffices to achieve all leave graphs for  $n = 12$ . The leave graph has two edges in this case. Hence, the possible leave graphs are  $P_3$  and  $2K_2$ . The leave graph  $2K_2$  was achieved by Roditty [39]. In order to achieve the leave graph  $P_3$ , write  $K_{12} = K_8 \vee K_4$  and label the vertices of  $K_8 \vee K_4$  with a  $(\mathbb{Z}_8, \mathbb{Z}_4)$ -labeling. Let  $R$  and  $S$  be as above. The complete bipartite graph with the partite sets  $\{3_1, 4_1, 5_1, 6_1\}$  and  $\{0_2, 1_2, 2_2, 3_2\}$  has a  $P_5$ -decomposition,  $T'$ , by Lemma 1.3.14. Let  $U$  consist of the single path  $(2_2, 0_2, 7_1, 3_2, 1_2)$ . Therefore,  $R \cup S \cup T' \cup U$  forms a maximum  $P_5$ -packing of  $K_{12}$  with the leave graph  $P_3$ . The edges of the leave graph are  $\{7_1, 1_2\}$  and  $\{7_1, 2_2\}$ .

Case 5.  $n \equiv 5 \pmod{8}$ ,  $T = A$

By Theorem 1.3.2, the leave graph has two edges in this case. The possible leave graphs are  $P_3$  and  $2K_2$ . By Corollary 1.3.15 it suffices to obtain all leave graphs for  $n = 13$ . Roditty showed how to achieve the leave graph  $P_3$  [39]. In order to obtain the leave graph  $2K_2$ , write  $K_{13} = K_{11} \vee K_2$ . Label the vertices of  $K_{11} \vee K_2$  with a  $(\mathbb{Z}_{11}, \mathbb{Z}_2)$ -labeling. In Case 3, we showed that there is a maximum  $A$ -packing of  $K_{11}$  with the leave graph  $3K_2$ . Let  $R$  be that packing and the edges of the leave graph be  $\{5_1, 6_1\}$ ,  $\{7_1, 8_1\}$ , and  $\{9_1, 10_1\}$ . Consider the complete bipartite graph with



partite sets  $\{0_1, 1_1, 2_1, \dots, 7_1\}$  and  $\{0_2, 1_2\}$ . This graph has an  $A$ -decomposition,  $S$ , by Lemma 1.3.14. Let  $U$  be formed by the trees  $(0_2; 8_1, 9_1, 10_1 - 1_2)$  and  $(1_2; 0_2, 8_1, 9_1 - 10_1)$ . Therefore,  $R \cup S \cup U$  forms a maximum  $A$ -packing of  $K_{13}$  with the leave graph  $2K_2$ . The edges of the leave graph are  $\{5_1, 6_1\}$  and  $\{7_1, 8_1\}$ .

Case 6.  $n \equiv 5 \pmod{8}$ ,  $T = P_5$

By Corollary 1.3.15, it suffices to achieve all leave graphs for  $n = 13$ . The leave graph has two edges in this case. Hence, the possible leave graphs are  $P_3$  and  $2K_2$ . The leave graph  $P_3$  was achieved by Roditty [39]. In order to achieve the leave graph  $2K_2$ , write  $K_{13} = K_9 \vee K_4$ . Label the vertices of  $K_9 \vee K_4$  with a  $(\mathbb{Z}_9, \mathbb{Z}_4)$ -labeling. Let  $R$  be a  $P_5$ -decomposition of  $K_9$ . The set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , the edges between these two sets, and the edges within the second set form a graph  $K_4 \vee 3K_1$ . The following paths form a maximum  $P_5$ -packing,  $S$ , of  $K_4 \vee 3K_1$  with the edges  $\{0_2, 2_2\}$  and  $\{1_2, 3_2\}$  as the leave graph which form a graph  $2K_2$ .

$$(1_2, 0_2, 0_1, 2_2, 1_1), (2_2, 1_2, 1_1, 3_2, 2_1), (3_2, 2_2, 2_1, 0_2, 1_1), (0_2, 3_2, 0_1, 1_2, 2_1)$$

The complete bipartite graph with one partite set  $\{3_1, 4_1, 5_1, 6_1, 7_1, 8_1\}$  and the other partite set  $\{0_2, 1_2, 2_2, 3_2\}$ , has a  $P_5$ -decomposition,  $U$ , by Lemma 1.3.14. Therefore,  $R \cup S \cup U$  forms a maximum  $P_5$ -packing of  $K_{13}$  with the leave graph  $2K_2$ . The edges of the leave graph are  $\{0_2, 2_2\}$  and  $\{1_2, 3_2\}$ .

Case 7.  $n \equiv 6 \pmod{8}$ ,  $T = A$

By Theorem 1.3.2, the leave graph has three edges in this case. The possible leave graphs are those mentioned in Case 1. By Corollary 1.3.15, it suffices to obtain all possible leave graphs for  $n = 14$ . Write  $K_{14} = K_{11} \vee K_3$ . Label the vertices of  $K_{11} \vee K_3$  with a  $(\mathbb{Z}_{11}, \mathbb{Z}_3)$ -labeling. Let  $H$  be any simple graph with three edges. By Case 1, there is a maximum  $A$ -packing of  $K_{11}$ ,  $R$ , with the leave graph  $H$ . The set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges between these two sets, and the edges within the second set, forms a graph  $K_3 \vee 3K_1$ . The following trees form an  $A$ -decomposition,  $S$ , of the graph  $K_3 \vee 3K_1$ .

$$(0_2; 0_1, 1_1, 1_2 - 2_1), (1_2; 0_1, 1_1, 2_2 - 2_1), (2_2; 0_1, 1_1, 0_2 - 2_1)$$

The complete bipartite graph with one partite set  $\{3_1, 4_1, 5_1, 6_1, 7_1, 8_1, 9_1, 10_1\}$  and the

other partite set  $\{0_2, 1_2, 2_2\}$ , has an  $A$ -decomposition,  $U$ , by Lemma 1.3.14. Therefore,  $R \cup S \cup U$  forms a maximum  $A$ -packing of  $K_{14}$  with the leave graph  $H$ . This completes the proof in this case.

Case 8.  $n \equiv 6 \pmod{8}$ ,  $T = P_5$

The leave graph has three edges in this case by Theorem 1.3.2. Hence, the possible leave graphs are those mentioned in Case 1. By Corollary 1.3.15, it suffices to achieve all possible leave graphs for  $n = 14$ . Let  $H$  be any possible leave graph with three edges. Write  $K_{14} = K_{11} \vee K_3$ . Label the vertices of  $K_{11} \vee K_3$  with a  $(\mathbb{Z}_{11}, \mathbb{Z}_3)$ -labeling. By Case 2, there exists a maximum  $P_5$ -packing,  $R$ , of  $K_{11}$  with the leave graph  $H$ . The set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges between these two sets, and the edges within the second set, form a graph  $K_3 \vee 3K_1$ . The following paths form a  $P_5$ -decomposition,  $S$ , of the graph  $K_3 \vee 3K_1$ .

$$(1_2, 0_2, 0_1, 2_2, 1_1), (2_2, 1_2, 1_1, 0_2, 2_1), (0_2, 2_2, 2_1, 1_2, 0_1)$$

The complete bipartite graph with partite sets  $\{3_1, 4_1, 5_1, \dots, 10_1\}$  and  $\{0_2, 1_2, 2_2\}$ , has a  $P_5$ -decomposition,  $U$ , by Lemma 1.3.14. Therefore,  $R \cup S \cup U$  forms a maximum  $P_5$ -packing of  $K_{14}$  with the leave graph  $H$ . ■

## 3.2 The Spectrum of Leave Graphs for Trees with Five Edges

**Theorem 3.2.1** *Let  $n \geq 9$  be an integer,  $T$  be any tree with five edges, and let the leave graph in a  $T$ -packing of the complete graph  $K_n$  have  $i$  edges. For any graph  $H$  with  $i$  edges there exists a maximum  $T$ -packing of  $K_n$  such that the leave graph is isomorphic to  $H$ .*

**Proof.** Let  $n \geq 9$  be an integer and  $T$  be any tree with five edges. As previously stated, the trees with five edges are  $S_5$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ , and  $P_6$ , as shown in Figure 3.3. The trees  $B$ ,  $C$ ,  $D$ , and  $E$ , are denoted by  $(x_1; x_2, x_3, x_4, x_5 - x_6)$ ,  $(x_1; x_2, x_3, x_4 - x_5 - x_6)$ ,  $(x_3; x_6, x_2, x_4 - x_1, x_5)$ , and  $(x_1 - x_2, x_3; x_4 - x_5, x_6)$ , respectively.

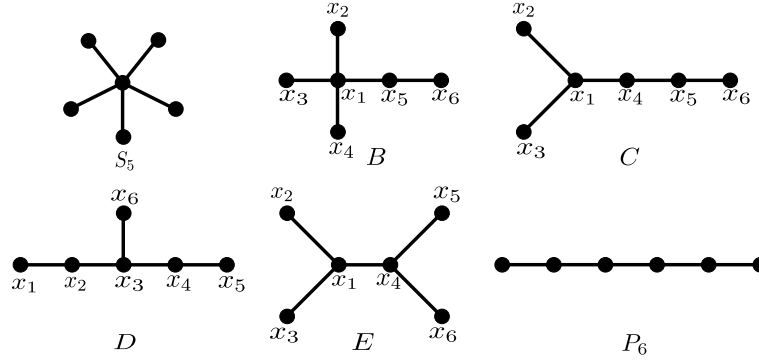


Figure 3.3: All trees with five edges

For the tree  $S_5$  the result is proved in Theorem 2.3.1. By Lemma 1.3.17, for each tree, it suffices to show the result for  $n = 9, 10, 11, 12, 13$ . For  $n = 10$  and  $11$  there is a  $T$ -decomposition of  $K_n$  by Theorem 1.3.1. For  $n = 9, 12$ , the leave graph has a single edge by Theorem 1.3.2. Therefore, we only need to achieve all possible leave graphs for  $n = 13$ . By Theorem 1.3.2, the leave graph in a maximum  $T$ -packing of  $K_{13}$  has three edges. Hence, the possible leave graphs are  $K_3$ ,  $S_3$ ,  $P_4$ ,  $P_3 + K_2$ , and  $3K_2$ . Now, for each tree  $T$ , we construct maximum  $T$ -packings with each of these leave graphs.

Case 1.  $T = B$

In order to obtain the leave graph  $K_3$ , write  $K_{13} = K_{10} \vee K_3$ . Label the vertices of  $K_{10} \vee K_3$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.1,  $K_{10}$  has a  $B$ -decomposition,  $R$ . Moreover, the complete bipartite graph with one partite set  $\{0_1, 1_1, 2_1, \dots, 9_1\}$  and the other partite set  $\{0_2, 1_2, 2_2\}$  has a  $B$ -decomposition,  $S$ , by Corollary 1.3.16. Let  $S$  consist of the following trees:

$$\begin{aligned} & (0_2; 0_1, 1_1, 2_1, 3_1 - 1_2), (1_2; 0_1, 1_1, 4_1, 2_1 - 2_2), (2_2; 0_1, 1_1, 3_1, 4_1 - 0_2), \\ & (0_2; 5_1, 6_1, 7_1, 8_1 - 1_2), (1_2; 5_1, 6_1, 9_1, 7_1 - 2_2), (2_2; 5_1, 6_1, 8_1, 9_1 - 0_2) \end{aligned}$$

Therefore,  $R \cup S$  is a maximum  $B$ -packing of  $K_{13}$  with the leave graph  $K_3$ . The edges of the leave graph are  $\{0_2, 1_2\}$ ,  $\{0_2, 2_2\}$ , and  $\{1_2, 2_2\}$ . We can obtain all the other possible leave graphs, except  $S_3$ , by making small changes to this construction. (See Table 3.3.)

New tree(s)	Previous tree(s)	Edges of the leave graph	Leave
$(0_2; 2_2, 1_1, 2_1, 3_1 - 1_2)$	$(0_2; 0_1, 1_1, 2_1, 3_1 - 1_2)$	$\{0_1, 0_2\}, \{0_2, 1_2\}, \{1_2, 2_2\}$	$P_4$
$(0_2; 2_2, 1_1, 2_1, 3_1 - 1_2)$ $(2_2; 1_2, 0_1, 1_1, 4_1 - 0_2)$	$(0_2; 0_1, 1_1, 2_1, 3_1 - 1_2)$ $(2_2; 0_1, 1_1, 3_1, 4_1 - 0_2)$	$\{0_1, 0_2\}, \{0_2, 1_2\}, \{3_1, 2_2\}$	$P_3 + K_2$
$(0_2; 2_2, 1_1, 2_1, 3_1 - 1_2),$ $(2_2; 1_2, 0_1, 1_1, 4_1 - 0_2),$ $(1_2; 0_2, 0_1, 4_1, 2_1 - 2_2)$	$(0_2; 0_1, 1_1, 2_1, 3_1 - 1_2),$ $(2_2; 0_1, 1_1, 3_1, 4_1 - 0_2),$ $(1_2; 0_1, 1_1, 4_1, 2_1 - 2_2)$	$\{0_1, 0_2\}, \{1_1, 1_2\}, \{3_1, 2_2\}$	$3K_2$

Table 3.3: Substitutions in the  $B$ -packing  $R \cup S$  to obtain different leave graphs for  $n = 13$

In order to obtain the leave graph  $S_3$ , consider the same partition and labeling of the vertices of  $K_{13}$  and let  $R$  be the same  $B$ -decomposirion of  $K_{10}$ . The complete bipartite graph with partite sets  $\{4_1, 5_1, 6_1, 7_1, 8_1\}$  and  $\{0_2, 1_2, 2_2\}$ , has a  $B$ -decomposition,  $S'$ , by Lemma 1.3.16. The set of vertices  $\{0_1, 1_1, 2_1, 3_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges between these two sets, and the edges within the second set, form a graph  $K_3 \vee 4K_1$ . The following trees form a  $B$ -decomposition,  $U$ , of  $K_3 \vee 4K_1$ .

$$(i_2; 0_1, 1_1, 2_1, (i + 1)_2 - 3_1), \quad i \in \mathbb{Z}_3$$

Therefore,  $R \cup S' \cup U$  forms a maximum  $B$ -packing of  $K_{13}$  with the leave graph  $S_3$ . In fact, the leave graph is the 3-star  $(9_1; 0_2, 1_2, 2_2)$ .

Case 2.  $T = C$

In order to obtain the leave graph  $K_3$ , write  $K_{13} = K_{10} \vee K_3$ . Label the vertices of  $K_{10} \vee K_3$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_3)$ -labeling. Let  $R$  be the  $C$ -decomposition of  $K_{10}$ . The following trees form a  $C$ -decomposition,  $S$ , of the bipartite graph with one part of vertices  $0_1, 1_1, 2_1, \dots, 9_1$  and the other part of vertices  $0_2, 1_2, 2_2$ .

$$(0_2; 0_1, 1_1, 2_1 - 1_2 - 3_1), (1_2; 0_1, 1_1, 4_1 - 2_2 - 2_1), (2_2; 0_1, 1_1, 3_1 - 0_2 - 4_1),$$

$$(0_2; 5_1, 6_1, 7_1 - 1_2 - 8_1), (1_2; 5_1, 6_1, 9_1 - 2_2 - 7_1), (2_2; 5_1, 6_1, 8_1 - 0_2 - 9_1)$$

Therefore,  $R \cup S$  forms a maximum  $C$ -packing of  $K_{13}$  with the leave graph  $K_3$ .

In order to obtain the leave graphs  $P_4$ ,  $P_3 + K_2$ , and  $3K_2$ , we substitute some trees in the packing  $R \cup S$  with new ones as shown in Table 3.4.

In order to achieve the leave graph  $S_3$ , partition and label the vertices of  $K_{13}$  as above. Let  $R$  be the  $C$ -decomposition of  $K_{10}$ . The set of vertices  $\{0_1, 1_1, 2_1, 3_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges between these two sets, and the edges within the

New tree(s)	Previous tree(s)	The edges of the leave graph	Leave
$(0_2; 2_2, 1_1, 2_1 - 1_2 - 3_1)$	$(0_2; 0_1, 1_1, 2_1 - 1_2 - 3_1)$	$\{0_1, 0_2\}, \{0_2, 1_2\}, \{1_2, 2_2\}$	$P_4$
$(0_2; 2_2, 1_1, 2_1 - 1_2 - 3_1)$ $(1_2; 0_1, 0_2, 4_1 - 2_2 - 2_1)$	$(0_2; 0_1, 1_1, 2_1 - 1_2 - 3_1)$ $(1_2; 0_1, 1_1, 4_1 - 2_2 - 2_1)$	$\{0_1, 0_2\}, \{1_1, 1_2\}, \{1_2, 2_2\}$	$P_3 + K_2$
$(0_2; 2_2, 1_1, 2_1 - 1_2 - 3_1)$ $(1_2; 0_1, 0_2, 4_1 - 2_2 - 2_1)$ $(2_2; 1_2, 6_1, 8_1 - 0_2 - 9_1)$	$(0_2; 0_1, 1_1, 2_1 - 1_2 - 3_1)$ $(1_2; 0_1, 1_1, 4_1 - 2_2 - 2_1)$ $(2_2; 5_1, 6_1, 8_1 - 0_2 - 9_1)$	$\{0_1, 0_2\}, \{1_1, 1_2\}, \{5_1, 2_2\}$	$3K_2$

Table 3.4: Substitutions in the  $C$ -packing  $R \cup S$  to obtain different leave graphs for  $n = 13$

second set, form a graph  $K_3 \vee 4K_1$ . The following trees form a  $C$ -decomposition,  $S'$ , of  $K_3 \vee 4K_1$ .

$$(0_2; 1_2, 0_1, 1_1 - 2_2 - 2_1), (1_2; 2_2, 0_1, 2_1 - 0_2 - 3_1), (2_2; 0_2, 0_1, 3_1 - 1_2 - 1_1)$$

The complete bipartite graph with partite sets  $\{4_1, 5_1, 6_1, 7_1, 8_1\}$  and  $\{0_2, 1_2, 2_2\}$ , have a  $C$ -decomposition,  $U$ , by Lemma 1.3.16. Therefore,  $R \cup S' \cup U$  forms a maximum  $C$ -packing of  $K_{13}$  with the leave graph  $S_3$ . In fact, the leave graph is the 3-star  $(9_1; 0_2, 1_2, 2_2)$ .

Case 3.  $T = D$

In order to achieve the leave graph  $K_3$ , write  $K_{13} = K_{10} \vee K_3$ . Label the vertices of  $K_{10} \vee K_3$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_3)$ -labeling. Let  $R$  be a  $D$ -decomposition of  $K_{10}$ . The graph  $K_{10,3}$  has a  $D$ -decomposition,  $S$ , by Lemma 1.3.16. Therefore,  $R \cup S$  forms a maximum  $D$ -packing of  $K_{13}$  with the leave graph  $K_3$ . The edges of the leave graph are  $\{0_2, 1_2\}, \{0_2, 2_2\}$ , and  $\{1_2, 2_2\}$ .

In order to obtain the leave graph  $S_3$ , partition and label the vertices of  $K_{13}$  as above. The set of vertices  $\{0_1, 1_1, 2_1, 3_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges between these two sets, and the edges within the second set, form a graph  $K_3 \vee 4K_1$ . The following graphs form a  $D$ -decomposition,  $S'$ , of the graph  $K_3 \vee 4K_1$ .

$$(0_2; 1_1, 0_1, 1_2 - 2_2, 2_1), (1_2; 0_1, 2_2, 3_1 - 2_1, 0_2), (2_2; 3_1, 0_2, 1_1 - 2_1, 1_2)$$

The complete bipartite graph with partite sets  $\{4_1, 5_1, 6_1, 7_1, 8_1\}$  and  $\{0_2, 1_2, 2_2\}$  has a  $D$ -decomposition,  $U$ , by Lemma 1.3.16. Therefore,  $R \cup S' \cup U$  forms a maximum  $D$ -packing of  $K_{13}$  with the 3-star  $(9_1; 0_2, 1_2, 2_2)$  as the leave graph.

Substitution of some trees  $D$  with some others in the packing  $R \cup S' \cup U$  leads to the leave graphs  $P_4$ ,  $P_3 + K_2$ , and  $3K_2$  (see Table 3.5).

New tree(s)	Previous tree(s)	The edges of the leave graph	Leave
$(9_1; 0_2, 1_2, 2_2 - 0_1, 2_1)$	$(1_2; 0_1, 2_2, 3_1 - 2_1, 0_2)$	$\{1_2, 2_2\}, \{3_1, 1_2\}, \{3_1, 0_2\}$	$P_4$
$(1_2; 9_1, 2_2, 3_1 - 2_1, 0_2)$	$(1_2; 0_1, 2_2, 3_1 - 2_1, 0_2)$	$\{0_1, 1_2\}, \{9_1, 0_2\}, \{9_1, 2_2\}$	$P_3 + K_2$
$(1_2; 9_1, 2_2, 3_1 - 2_1, 0_2)$ $(0_2; 9_1, 1_2, 0_1 - 2_1, 2_2)$	$(1_2; 0_1, 2_2, 3_1 - 2_1, 0_2)$ $(0_2; 1_1, 1_2, 0_1 - 2_1, 2_2)$	$\{0_1, 1_2\}, \{0_2, 1_1\}, \{9_1, 2_2\}$	$3K_2$

Table 3.5: Substitutions in the  $D$ -packing  $R \cup S' \cup U$  to obtain different leave graphs for  $n = 13$

Case 4.  $T = E$

In order to obtain the leave graph  $K_3$ , write  $K_{13} = K_{10} \vee K_3$ . Label the vertices of  $K_{10} \vee K_3$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_3)$ -labeling. The graph  $K_{10}$  has an  $E$ -decomposition,  $R$ . Let  $R$  contain the following trees. Note that the addition is taken modulo 9.

$$(i_1 - 9_1, (i + 1)_1; (i + 2)_1 - (i + 5)_1, (i + 6)_1), i \in \mathbb{Z}_9$$

By Lemma 1.3.17, the graph  $K_{3,10}$  has an  $E$ -decomposition,  $S$ . Therefore,  $R \cup S$  forms a maximum  $E$ -packing of  $K_{13}$  with the leave graph  $K_3$ . The edges of the leave graph are  $\{0_2, 1_2\}$ ,  $\{0_2, 2_2\}$ , and  $\{1_2, 2_2\}$ .

In order to obtain the leave graph  $S_3$ , partition and label the vertices of  $K_{13}$  as above. The set of vertices  $\{0_1, 1_1, 2_1, 3_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , the edges between these two sets, and the edges within the second set, form a graph  $K_3 \vee 4K_1$ . The following trees form an  $E$ -decomposition,  $U$ , of the graph  $K_3 \vee 4K_1$ .

$$(0_2 - 0_1, 1_1; 1_2 - 2_1, 3_1), (1_2 - 0_1, 1_1; 2_2 - 2_1, 3_1), (2_2 - 0_1, 1_1; 0_2 - 2_1, 3_1)$$

The vertices  $4_1, 5_1, 6_1, 7_1, 8_1$ , the vertices  $0_2, 1_2, 2_2$ , and the edges between them, form a graph  $K_{3,5}$ , which has an  $E$ -decomposition,  $V$ , by Lemma 1.3.16. Therefore,  $R \cup U \cup V$  forms a maximum  $E$ -packing of  $K_{13}$  with the 3-star  $(9_1; 0_2, 1_2, 2_2)$  as the leave graph.

In order to obtain the remaining leave graphs, we substitute some trees for others in the packing  $R \cup U \cup V$ . Table 3.6 illustrates these substitutions.

New tree(s)	Previous tree(s)	Edges of the leave graph	Leave
$(0_2 - 0_1, 1_1; 9_1 - 1_2, 2_2)$ $(2_1 - 9_1, 1_2; 4_1 - 7_1, 8_1)$	$(0_2 - 0_1, 1_1; 1_2 - 2_1, 3_1)$ $(2_1 - 9_1, 3_1; 4_1 - 7_1, 8_1)$	$\{2_1, 3_1\}, \{3_1, 1_2\}, \{0_2, 1_2\}$	$P_4$
$(0_2 - 1_1, 9_1; 1_2 - 2_1, 3_1)$	$(0_2 - 0_1, 1_1; 1_2 - 2_1, 3_1)$	$\{0_1, 0_2\}, \{9_1, 1_2\}, \{9_1, 2_2\}$	$P_3 + K_2$
$(0_2 - 1_1, 9_1; 1_2 - 2_1, 3_1)$ $(1_2 - 0_1, 9_1; 2_2 - 2_1, 3_1)$	$(0_2 - 0_1, 1_1; 1_2 - 2_1, 3_1)$ $(1_2 - 0_1, 1_1; 2_2 - 2_1, 3_1)$	$\{0_1, 0_2\}, \{1_1, 1_2\}, \{9_1, 2_2\}$	$3K_2$

Table 3.6: Substitutions in the  $E$ -packing  $R \cup U \cup V$  to obtain different leave graphs for  $n = 13$

Case 5.  $T = P_6$

In order to obtain the leave graph  $K_3$ , write  $K_{13} = K_9 \vee K_4$ . Label the vertices of  $K_9 \vee K_4$  with a  $(\mathbb{Z}_9, \mathbb{Z}_4)$ -labeling. The graph  $K_9$  has a maximum  $P_6$ -packing,  $R$ , with a single edge as the leave graph. Let the leave graph be the edge  $\{7_1, 8_1\}$ . By Parker's theorem [32], the complete bipartite graph with partite sets  $\{0_1, 1_1, 2_1, 3_1, 4_1\}$  and  $\{0_2, 1_2, 2_2, 3_2\}$  has a  $P_6$ -decomposition,  $S$ . The set of vertices  $\{5_1, 6_1, 7_1, 8_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , the edges between them, and the edges within the second set, form a graph  $K_4 \vee 4K_1$ . The following paths form a maximum  $P_6$ -packing,  $U$ , of the graph  $K_4 \vee 4K_1 \cup \{7_1, 8_1\}$  with the leave graph  $K_3$ .

$$(5_1, 0_2, 6_1, 1_2, 7_1, 8_1), (5_1, 1_2, 0_2, 2_2, 7_1, 3_2),$$

$$(7_1, 0_2, 3_2, 6_1, 2_2, 1_2), (0_2, 8_1, 1_2, 3_2, 5_1, 2_2)$$

Therefore,  $R \cup S \cup U$  forms a maximum  $P_6$ -packing of  $K_{13}$  with the leave graph  $K_3$ . The edges of the leave graph are  $\{8_1, 2_2\}, \{8_1, 3_2\}$ , and  $\{2_2, 3_2\}$ . Table 3.7 demonstrates the substitutions needed in the packing  $R \cup S \cup U$  in order to obtain the other leave graphs.

New path(s)	Previous path(s)	Edges of the leave graph	Leave
$(5_1, 0_2, 6_1, 1_2, 7_1, 2_2)$ $(5_1, 1_2, 0_2, 2_2, 3_2, 7_1)$	$(5_1, 0_2, 6_1, 1_2, 7_1, 8_1)$ $(5_1, 1_2, 0_2, 2_2, 7_1, 3_2)$	$\{7_1, 8_1\}, \{8_1, 2_2\}, \{8_1, 3_2\}$	$S_3$
$(5_1, 1_2, 0_2, 2_2, 3_2, 8_1)$	$(5_1, 1_2, 0_2, 2_2, 7_1, 3_2)$	$\{8_1, 2_2\}, \{7_1, 2_2\}, \{7_1, 3_2\}$	$P_4$
$(0_2, 6_1, 1_2, 7_1, 8_1, 3_2)$	$(5_1, 0_2, 6_1, 1_2, 7_1, 8_1)$	$\{5_1, 0_2\}, \{8_1, 2_2\}, \{2_2, 3_2\}$	$P_3 + K_2$
$(1_2, 0_2, 2_2, 3_2, 8_1, 7_1)$ $(5_1, 0_2, 6_1, 1_2, 7_1, 2_2)$	$(5_1, 1_2, 0_2, 2_2, 7_1, 3_2)$ $(5_1, 0_2, 6_1, 1_2, 7_1, 8_1)$	$\{5_1, 1_2\}, \{7_1, 3_2\}, \{8_1, 2_2\}$	$3K_2$

Table 3.7: Substitutions in the  $P_6$ -packing  $R \cup S \cup U$  to obtain different leave graphs for  $n = 13$

■

## Chapter 4

# The Spectrum of Excess Graphs for Trees with up to Five Edges

This chapter will discuss the spectrum of excess graphs for trees with up to five edges. For trees with one edge, the decomposition always exists. For trees with two edges, the excess graph has at most one edge [37], and the only possible excess graph will be  $K_2$ .

### 4.1 The Spectrum of Excess Graphs for Trees with Three Edges

In this section, we will find the spectrum of excess graphs for trees with three edges.

**Theorem 4.1.1** *Let  $T$  be any tree with three edges and  $n \geq 6$ . If the excess graph in a minimum  $T$ -covering of  $K_n$  has  $i$  edges and  $E$  is any multigraph with  $i$  edges, then there exists a minimum  $T$ -covering of  $K_n$  with the excess graph  $E$ .*

**Proof.** Let  $n$  be any positive integer such that  $n \geq 6$ . The only trees with three edges are  $S_3$  and  $P_4$ .

By Lemma 1.3.10, we only need to prove the theorem for  $n = 6, 7$ , and  $8$ . The tree  $S_3$  has been considered in Chapter 2. Consider  $P_4$ . By Lemma 1.3.13, it suffices



to find the spectrum of excess graphs for  $P_4$ , in the cases where  $n = 6, 7, 8$ . According to Theorem 1.3.1,  $K_6$  and  $K_7$  have  $P_4$ -decompositions.

For  $n = 8$ , by Theorem 1.3.3, the excess graph in a minimum  $P_4$ -covering of  $K_8$  has two edges. Hence, the possible excess graphs are  $P_3$ ,  $2K_2$ , and  $K_2^2$ . Label the vertices of  $K_8$  with the elements of  $\mathbb{Z}_8$ . Let  $R$  be a maximum  $P_4$ -packing of  $K_8$  with the edge  $\{0, 1\}$  as the leave graph. Also let  $S$  be the set consisting of the single path  $(0, 1, 2, 3)$  and  $U$  be the set consisting of the single path  $(7, 0, 1, 2)$ . Therefore,  $R \cup S$  and  $R \cup U$  form  $P_4$ -coverings for  $K_8$  with the excess graphs  $P_3$  (with the edges  $\{1, 2\}$  and  $\{2, 3\}$ ) and  $2K_2$  (with the edges  $\{0, 7\}$  and  $\{1, 2\}$ ), respectively.

In order to obtain the excess graph  $K_2^2$ , write  $K_8 = K_5 \vee K_3$ . Label the vertices of  $K_5 \vee K_3$  with a  $(\mathbb{Z}_5, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.2, the leave graph in a maximum  $P_4$ -packing of  $K_5$  has one edge. Let  $R$  be a maximum  $P_4$ -packing of  $K_5$  with the edge  $\{3_1, 4_1\}$  as the leave graph. Consider  $S$  as the set consisting of the following paths.

$$\begin{aligned} &(0_1, 0_2, 1_2, 1_1), (0_1, 1_2, 0_2, 1_1), (2_1, 0_2, 1_2, 3_1), (0_1, 2_2, 1_2, 2_1), \\ &(1_1, 2_2, 0_2, 3_1), (2_1, 2_2, 4_1, 0_2), (2_2, 3_1, 4_1, 1_2) \end{aligned}$$

Therefore,  $R \cup S$  forms a minimum  $P_4$ -covering of  $K_8$  with the excess graph  $K_2^2$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used twice. ■

## 4.2 The Spectrum of Excess Graphs for Trees with Four Edges

**Theorem 4.2.1** *Let  $T$  be any tree with four edges and  $n \geq 8$ . If the excess graph in a minimum  $T$ -covering of  $K_n$  has  $i$  edges and  $E$  is any multigraph with  $i$  edges, then there exists a minimum  $T$ -covering of  $K_n$  with the excess graph  $E$ .*

**Proof.** Let  $n$  be any positive integer such that  $n \geq 8$ . The trees with four edges are  $S_4$ ,  $A$ , and  $P_5$ , where  $A$  is a 3-star with one edge joined to one of its end vertices. The corresponding problem for  $S_4$  was solved in Chapter 2. Now consider  $A$ . By Corollary 1.3.15, it suffices to achieve all possible excess graphs for  $n = 8, 9, 10, 11, 12, 13, 14, 15$ . For  $n = 8, 9$ , the complete graph  $K_n$  has an  $A$ -decomposition by Theorem 1.3.1.

Case 1(a).  $n = 10$ .

The excess graph has three edges in this case, by Theorem 1.3.3. The possible excess graphs with three edges are  $S_3, K_3, P_4, P_3 + K_2, 3K_2, K_2^3, F$ , and  $K_2^2 + K_2$ , where  $F$  is a  $K_2^2$  with an edge attached to one of its vertices.

In order to achieve the excess graphs  $S_3, P_4$ , and  $P_3 + K_2$ , label the vertices of  $K_{10}$  with the elements of  $\mathbb{Z}_{10}$ . By Theorem 1.3.2, the leave graph in a maximum  $A$ -packing of  $K_{10}$  has one edge. Let  $R$  be a maximum  $A$ -packing of  $K_{10}$  with the edge  $\{0, 1\}$  as the leave graph. Also let  $S$  be the set consisting of the single tree  $(7; 8, 9, 0 - 1)$ ,  $U$  be the set consisting of the single tree  $(1; 0, 2, 3 - 4)$ , and  $V$  be the set consisting of the single tree  $(0; 8, 9, 1 - 2)$ . Therefore,  $R \cup S$ ,  $R \cup U$ , and  $R \cup V$ , form minimum  $A$ -coverings of  $K_{10}$  with the excess graphs  $S_3$  (with the edges  $\{0, 7\}, \{7, 8\}$ , and  $\{7, 9\}$ ),  $P_4$  (with the edges  $\{1, 2\}, \{1, 3\}$ , and  $\{3, 4\}$ ), and  $P_3 + K_2$  (with the edges  $\{0, 8\}, \{0, 9\}$ , and  $\{1, 2\}$ ), respectively.

In order to obtain the excess graph  $K_2^3$ , write  $K_{10} = K_7 \vee K_3$  and label the vertices of  $K_7 \vee K_3$  with a  $(\mathbb{Z}_7, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.2, the leave graph in a maximum  $A$ -packing of  $K_7$  has one edge. Let  $R$  be a maximum  $A$ -packing of  $K_7$  with the edge  $\{5_1, 6_1\}$  as the leave graph, and  $S$  be the set consisting the following trees.

$$\begin{aligned} &(0_2; 1_2, 0_1, 1_1 - 2_2), (1_2; 0_2, 1_1, 0_1 - 2_2), (0_2; 1_2, 2_1, 5_1 - 2_2), \\ &(1_2; 0_2, 5_1, 2_1 - 2_2), (0_2; 4_1, 6_1, 3_1 - 1_2), (2_2; 3_1, 4_1, 6_1 - 5_1), \\ &(1_2; 4_1, 6_1, 2_2 - 0_2) \end{aligned}$$

Hence,  $R \cup S$  forms a minimum  $A$ -covering of  $K_{10}$  with the excess graph  $K_2^3$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used three times.

The remaining excess graphs will be obtained by substituting some trees for other trees in the covering  $R \cup S$ . Table 4.1 illustrates these substitutions.

Case 1(b).  $n = 11$ .

The excess graph has one edge by Theorem 1.3.3. Hence, the only possible excess graph is  $K_2$ . The minimum covering with this excess graph can be achieved easily by adding one tree  $A$  to a maximum  $A$ -packing of  $K_{11}$  with the leave graph  $S_3$ .

Case 1(c).  $n = 12$ .

By Theorem 1.3.3, the excess graph has two edges. Hence, the possible excess graphs are  $K_2^2, P_3$ , and  $2K_2$ . All leave graphs in  $A$ -packings of  $K_{12}$  are achievable as

New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
$(1_2; 0_1, 5_1, 2_1 - 2_2)$	$(1_2; 0_2, 5_1, 2_1 - 2_2)$	$\{0_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$F$
$(1_2; 4_1, 6_1, 2_2 - 5_1)$ $(0_2; 2_2, 2_1, 5_1 - 6_1)$	$(1_2; 4_1, 6_1, 2_2 - 0_2)$ $(0_2; 1_2, 2_1, 5_1 - 2_2)$	$\{5_1, 6_1\}, \{0_2, 1_2\}$ (twice)	$K_2^2 + K_2$
$(1_2; 0_1, 5_1, 2_1 - 2_2)$ $(0_2; 0_1, 2_1, 5_1 - 2_2)$	$(1_2; 0_2, 5_1, 2_1 - 2_2)$ $(0_2; 1_2, 2_1, 5_1 - 2_2)$	$\{0_1, 0_2\}, \{0_1, 1_2\}, \{0_2, 1_2\}$	$K_3$
$(1_2; 4_1, 6_1, 2_2 - 5_1)$ $(0_2; 2_2, 2_1, 5_1 - 6_1)$ $(0_2; 0_1, 2_1, 1_1 - 2_2)$ $(1_2; 1_1, 3_1, 0_1 - 2_2)$	$(1_2; 4_1, 6_1, 2_2 - 0_2)$ $(0_2; 1_2, 2_1, 5_1 - 2_2)$ $(0_2; 1_2, 0_1, 1_1 - 2_2)$ $(1_2; 0_2, 1_1, 0_1 - 2_2)$	$\{5_1, 6_1\}, \{0_1, 0_2\}, \{3_1, 1_2\}$	$3K_2$

Table 4.1: Substitutions in the  $A$ -covering  $R \cup S$  to obtain different excess graphs for  $n = 10$

we saw in Chapter 3. Hence, there is a maximum  $A$ -packing of  $K_{12}$  with the leave graph  $P_3$ . Label the vertices of  $K_{12}$  with the elements of  $\mathbb{Z}_{12}$  and let  $R$  be a maximum  $A$ -packing of  $K_{12}$  with the leave graph  $(0, 1, 2)$ . Also let  $S$  be the set consisting of the single tree  $(1; 0, 2, 3 - 4)$  and  $U$  be the set consisting of the single tree  $(1; 0, 3, 2 - 4)$ . Therefore,  $R \cup S$  and  $R \cup U$  are minimum  $A$ -coverings of  $K_{12}$  with the excess graphs  $P_3$  (with the edges  $\{1, 3\}$  and  $\{3, 4\}$ ) and  $2K_2$  (with the edges  $\{1, 3\}$  and  $\{2, 4\}$ ).

In order to obtain  $K_2^2$  as the excess graph, write  $K_{12} = K_8 \vee K_4$ . Label the vertices of  $K_8 \vee K_4$  with a  $(\mathbb{Z}_8, \mathbb{Z}_4)$ -labeling. By Theorem 1.3.1,  $K_8$  has an  $A$ -decomposition,  $R$ . Moreover, the set of vertices  $\{3_1, 4_1, 5_1, 6_1, 7_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , and the edges between these two sets, form a graph  $K_{5,4}$ , which has an  $A$ -decomposition,  $S$ , by Lemma 1.3.14. Also the set of vertices  $\{0_1, 1_1, 2_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , the edges between these two sets, and the edges within the latter, form a graph  $K_4 \vee 3K_1$ . The following trees form a minimum  $A$ -covering,  $U$ , of  $K_4 \vee 3K_1$  with the excess graph  $K_2^2$ .

$$(0_2; 1_2, 0_1, 1_1 - 2_2), (1_2; 0_2, 1_1, 2_1 - 2_2), (0_2; 2_2, 3_2, 1_2 - 0_1), \\ (3_2; 1_2, 0_1, 2_1 - 0_2), (2_2; 1_2, 0_1, 3_2 - 1_1)$$

Therefore,  $R \cup S \cup U$  forms a minimum  $A$ -covering of  $K_{12}$  with the excess graph  $K_2^2$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used twice.

Case 1(d).  $n = 13$ .

By Theorem 1.3.3, the excess graph has two edges. Hence, the possible excess graphs are  $K_2^2$ ,  $P_3$ , and  $2K_2$ . The excess graphs  $P_3$  and  $2K_2$  are achievable by the same argument about those excess graphs in Case 3.

In order to achieve the excess graph  $K_2^2$ , write  $K_{13} = K_9 \vee K_4$ . Label the vertices of  $K_9 \vee K_4$  with a  $(\mathbb{Z}_9, \mathbb{Z}_4)$ -labeling. By Theorem 1.3.1,  $K_9$  has an  $A$ -decomposition  $R$ . Furthermore, the set of vertices  $\{3_1, 4_1, 5_1, 6_1, 7_1, 8_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , and the edges between these two sets, form a graph  $K_{6,4}$ , which has an  $A$ -decomposition,  $S$ , by Lemma 1.3.14. Also let  $U$  be the minimum  $A$ -covering introduced in Case 3. Therefore,  $R \cup S \cup U$  forms a minimum  $A$ -covering of  $K_{13}$  with the excess graph  $K_2^2$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used twice.

Case 1(e).  $n = 14$ .

The excess graph has one edge by Theorem 1.3.3. Hence, the only possible excess graph is  $K_2$ . The minimum covering with this excess graph can be achieved easily by adding one tree  $A$  to a maximum  $A$ -packing of  $K_{14}$  with the leave graph  $S_3$ .

Case 1(f).  $n = 15$ .

The excess graph has three edges by Theorem 1.3.3. Hence, the possible excess graphs are  $S_3, K_3, P_4, P_3 + K_2, 3K_2, K_2^3, F$ , and  $K_2^2 + K_2$ . We will use Case 1 to achieve all these excess graphs at once. Write  $K_{15} = K_{10} \vee K_5$ . Label the vertices of  $K_{10} \vee K_5$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_5)$ -labeling. Let  $H$  be any of the possible excess graphs. In Case 1, we proved that all of these excess graphs are achievable in  $A$ -coverings of  $K_{10}$ . Let  $R$  be a minimum  $A$ -covering of  $K_{10}$  with the excess graph  $H$ . Moreover, the set of vertices  $\{2_1, 3_1, 4_1, \dots, 9_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2, 4_2\}$ , and the edges between these two sets, form a graph  $K_{5,8}$ , which has an  $A$ -decomposition,  $S$ , by Lemma 1.3.14. Now, consider the graph  $K_5 \vee 2K_1$ , formed by the set of vertices  $\{0_1, 1_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2, 4_2\}$ , the edges between these two sets, and the edges within the latter. The following trees form an  $A$ -decomposition,  $U$ , of  $K_5 \vee 2K_1$ .

$$(i_2; 0_1, 1_1, (i+1)_2 - (i+3)_2), i \in \mathbb{Z}_5$$

Therefore,  $R \cup S \cup U$  forms a minimum  $A$ -covering of  $K_{15}$  with the excess graph  $H$ .

Now consider  $P_5$ . By Corollary 1.3.15, it suffices to prove that all excess graphs are achievable in  $P_5$ -coverings of  $K_n$  for  $n = 8, 9, 10, 11, 12, 13, 14, 15$ . For  $n = 8, 9$ ,  $K_n$  has a  $P_5$ -decomposition by Theorem 1.3.1. For the remaining cases, we show how we can achieve all possible excess graphs.

Case 2(a).  $n = 10$

The excess graph has three edges by Theorem 1.3.3. Therefore, the possible excess graphs are  $S_3, K_3, P_4, P_3 + K_2, 3K_2, K_2^3, F$ , and  $K_2^2 + K_2$ . The excess graphs  $P_4$  and  $P_3 + K_2$  can be obtained by adding one path to a maximum packing. In fact, by Theorem 1.3.2, the leave graph in a maximum  $P_5$ -packing of  $K_{10}$  has one edge. Label the vertices of  $K_{10}$  with the elements of  $\mathbb{Z}_{10}$  and let  $R$  be a maximum  $P_5$ -packing of  $K_{10}$  with the edge  $\{0, 1\}$  as the leave graph. Also let  $S$  be the single path  $(0, 1, 2, 3, 4)$  and  $U$  be the single path  $(9, 0, 1, 2, 3)$ . Therefore,  $R \cup S$  and  $R \cup U$  form minimum  $P_5$ -coverings of  $K_{10}$  with the excess graphs  $P_4$  (with the edges  $\{1, 2\}, \{2, 3\}$ , and  $\{3, 4\}$ ) and  $P_3 + K_2$  (with the edges  $\{0, 9\}, \{1, 2\}$ , and  $\{2, 3\}$ ), respectively.

In order to achieve the excess graph  $K_2^3$ , write  $K_{10} = K_7 \vee K_3$  and label the vertices of  $K_7 \vee K_3$  with a  $(\mathbb{Z}_7, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.2, the leave graph in a maximum  $P_5$ -packing of  $K_7$  has one edge. Let  $R$  be a maximum  $P_5$ -packing of  $K_7$  with the edge  $\{5_1, 6_1\}$  as the leave graph. Also let  $S$  be the set consisting of the following paths.

$$\begin{aligned} &(0_1, 0_2, 1_2, 1_1, 2_2), (1_1, 0_2, 1_2, 0_1, 2_2), (2_1, 0_2, 1_2, 3_1, 2_2), \\ &(3_1, 0_2, 1_2, 2_1, 2_2), (5_1, 2_2, 0_2, 6_1, 1_2), (4_1, 0_2, 5_1, 1_2, 2_2), \\ &(5_1, 6_1, 2_2, 4_1, 1_2) \end{aligned}$$

Therefore,  $R \cup S$  is a minimum  $P_5$ -covering of  $K_{10}$  with the excess graph  $K_2^3$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used three times.

The rest of the excess graphs will be achieved by substituting some paths for some others in the minimum covering  $R \cup S$ . These substitutions are illustrated in Table 4.2.

Case 2(b).  $n = 11$

The excess graph has one edge by Theorem 1.3.3. The excess graph  $K_2$  can be achieved by adding one path to a maximum  $P_5$ -packing of  $K_{11}$  with the leave graph  $P_4$ .

Case 2(c).  $n = 12$

By Theorem 1.3.3, the excess graph has two edges. Hence, the possible excess graphs are  $K_2^2, P_3$ , and  $2K_2$ . We first obtain the excess graphs  $P_3$  and  $2K_2$ . All leave graphs in  $P_5$ -packings of  $K_{12}$  are achievable as illustrated in Chapter 3. Label the vertices of  $K_{12}$  with the elements of  $\mathbb{Z}_{12}$  and let  $R$  be a maximum  $P_5$ -packing of

New path(s)	Previous path(s)	Edges of the excess graph	Excess
$(1_1, 0_2, 2_2, 0_1, 1_2)$	$(1_1, 0_2, 1_2, 0_1, 2_2)$	$\{0_2, 1_2\}$ (twice), $\{0_2, 2_2\}$	$F$
$(1_1, 0_2, 2_2, 0_1, 1_2)$ $(0_1, 0_2, 2_2, 1_1, 1_2)$ $(0_2, 3_1, 1_2, 2_1, 2_2)$	$(1_1, 0_2, 1_2, 0_1, 2_2)$ $(0_1, 0_2, 1_2, 1_1, 2_2)$ $(3_1, 0_2, 1_2, 2_1, 2_2)$	$\{0_2, 2_2\}$ (twice), $\{3_1, 1_2\}$	$K_2^2 + K_2$
$(2_2, 0_1, 1_2, 1_1, 0_2)$ $(2_2, 3_1, 1_2, 2_1, 0_2)$	$(2_2, 0_1, 1_2, 0_2, 1_1)$ $(2_2, 3_1, 1_2, 0_2, 2_1)$	$\{1_1, 1_2\}, \{2_1, 1_2\}, \{0_2, 1_2\}$	$S_3$
$(3_1, 0_2, 2_2, 2_1, 1_2)$ $(1_1, 0_2, 2_2, 0_1, 1_2)$ $(5_1, 2_2, 1_2, 6_1, 0_2)$	$(3_1, 0_2, 1_2, 2_1, 2_2)$ $(1_1, 0_2, 1_2, 0_1, 2_2)$ $(5_1, 2_2, 0_2, 6_1, 1_2)$	$\{0_2, 1_2\}, \{0_2, 2_2\}, \{1_2, 2_2\}$	$K_3$
$(3_1, 4_1, 0_2, 5_1, 1_2)$ $(5_1, 2_2, 1_2, 6_1, 0_2)$ $(3_1, 0_2, 2_2, 2_1, 1_2)$ $(1_2, 3_1, 2_2, 2_1, 0_2)$ $(2_2, 0_1, 1_2, 1_1, 0_2)$	$(4_1, 0_2, 5_1, 1_2, 2_2)$ $(5_1, 2_2, 0_2, 6_1, 1_2)$ $(3_1, 0_2, 1_2, 2_1, 2_2)$ $(2_1, 0_2, 1_2, 3_1, 2_2)$ $(2_2, 0_1, 1_2, 0_2, 1_1)$	$\{3_1, 4_1\}, \{1_1, 1_2\}, \{2_1, 2_2\}$	$3K_2$

Table 4.2: Substitutions in the  $P_5$ -covering  $R \cup S$  to obtain different excess graphs for  $n = 10$

$K_{12}$  with the path  $(0, 1, 2)$  as the leave graph. Let  $S$  be the single path  $(0, 1, 2, 3, 4)$  and  $U$  be the single path  $(11, 0, 1, 2, 3)$ . Therefore,  $R \cup S$  and  $R \cup U$  are minimum  $P_5$ -coverings of  $K_{12}$  with the excess graphs  $P_3$  (with the edges  $\{2, 3\}$  and  $\{3, 4\}$ ) and  $2K_2$  (with the edges  $\{0, 11\}$  and  $\{2, 3\}$ ), respectively.

In order to achieve the excess graph  $K_2^2$ , write  $K_{12} = K_8 \vee K_4$ . Label the vertices of  $K_8 \vee K_4$  with a  $(\mathbb{Z}_8, \mathbb{Z}_4)$ -labeling. By Theorem 1.3.1,  $K_8$  has a  $P_5$ -decomposition,  $R$ . Moreover, the set of vertices  $\{2_1, 3_1, 4_1, 5_1, 6_1, 7_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , and the edges between these two sets, form a graph  $K_{6,4}$ , which has a  $P_5$ -decomposition,  $S$ , by Lemma 1.3.14. Also the set of vertices  $\{0_1, 1_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , the edges between these two sets, and the edges within the latter set, form a graph  $K_4 \vee 2K_1$ . The following paths form a minimum  $P_5$ -covering,  $U$ , of  $K_4 \vee 2K_1$  with the excess graph  $K_2^2$ .

$$\begin{aligned}
&(0_1, 0_2, 1_2, 1_1, 2_2), (0_2, 1_2, 0_1, 2_2, 3_2), \\
&(0_1, 3_2, 0_2, 1_2, 2_2), (1_2, 3_2, 1_1, 0_2, 2_2)
\end{aligned}$$

Therefore,  $R \cup S \cup U$  forms a minimum  $P_5$ -covering of  $K_{12}$  with the excess graph  $K_2^2$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used twice.

Case 2(d).  $n = 13$

By Theorem 1.3.3, the excess graph has two edges. Hence, the possible excess

graphs are  $K_2^2, P_3$ , and  $2K_2$ . We first obtain the excess graphs  $P_3$  and  $2K_2$ . The excess graphs  $P_3$  and  $2K_2$  can be obtained from a maximum  $P_5$ -packing of  $K_{13}$  in a similar way as explained in Case 3.

In order to obtain the excess graph  $K_2^2$ , write  $K_{13} = K_9 \vee K_4$  and label the vertices of  $K_9 \vee K_4$  with a  $(\mathbb{Z}_9, \mathbb{Z}_4)$ -labeling. By Theorem 1.3.1,  $K_9$  has a  $P_5$ -decomposition,  $R$ . Moreover, the set of vertices  $\{2_1, 3_1, \dots, 8_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , and the edges between these two sets, form a graph  $K_{7,4}$ , which has a  $P_5$ -decomposition,  $S$ , by Lemma 1.3.14. Also consider  $U$  as described in Case 3. Therefore,  $R \cup S \cup U$  forms a minimum  $P_5$ -covering of  $K_{13}$  with the excess graph  $K_2^2$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used twice.

Case 2(e).  $n = 14$

The excess graph has one edge by Theorem 1.3.3. The excess graph  $K_2$  can be achieved by adding one path to any maximum  $P_5$ -packing of  $K_{14}$  with the leave graph  $P_4$ .

Case 2(f).  $n = 15$

The excess graph has three edges by Theorem 1.3.3. Therefore, the possible excess graphs are  $S_3, K_3, P_4, P_3 + K_2, 3K_2, K_2^3, F$ , and  $K_2^2 + K_2$ . We will achieve all these excess graphs using Case 1. Write  $K_{15} = K_{10} \vee K_5$  and label the vertices of  $K_{10} \vee K_5$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_5)$ -labeling. Let  $H$  be any of the possible mentioned excess graphs and  $R$  be a minimum  $P_5$ -covering of  $K_{10}$  with the excess graph  $H$ . The set of vertices  $\{2_1, 3_1, \dots, 9_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2, 4_2\}$ , and the edges between these two sets, form a graph  $K_{8,5}$ , which has a  $P_5$ -decomposition,  $S$ , by Lemma 1.3.14. Also the set of vertices  $\{0_1, 1_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2, 4_2\}$ , the edges between these two sets, and the edges within the latter set, form a graph  $K_5 \vee 2K_1$ . The following paths form a  $P_5$ -decomposition,  $U$ , of  $K_5 \vee 2K_1$ .

$$(0_1, i_2, (i+1)_2, (i+3)_2, 1_1), i \in \mathbb{Z}_5$$

Therefore,  $R \cup S \cup U$  is a minimum  $P_5$ -covering of  $K_{15}$  with the excess graph  $H$ . ■

### 4.3 The Spectrum of Excess Graphs for Trees with Five Edges

**Theorem 4.3.1** *Let  $T$  be any tree with five edges and  $n \geq 10$ . If the excess graph in a minimum  $T$ -covering of  $K_n$  has  $i$  edges and  $E$  is any multigraph with  $i$  edges, then there exists a minimum  $T$ -covering of  $K_n$  with the excess graph  $E$ , except for the excess graph  $K_2^4$  which is not achievable when  $T = S_5$  and  $n = 12$ .*

**Proof.** Let  $n \geq 10$  be an integer and  $T$  any tree with five edges. For the case  $T = S_5$ , all excess graphs were achieved as illustrated in Chapter 2, except for the graph  $K_2^4$  which cannot be obtained as the excess graph in any  $S_5$ -covering of  $K_{12}$ . For the rest of the trees, it suffices to achieve all possible excess graphs for  $n = 10, 11, 12, 13, 14$  by Lemma 1.3.17. Furthermore, for  $n = 10, 11$  the complete graph  $K_n$  has a  $T$ -decomposition by Theorem 1.3.1. Now, we will prove that for  $n = 12, 13, 14$  all possible excess graphs are achievable.

Case 1.  $T = B, n = 12$

The excess graph has four edges by Theorem 1.3.3. Hence, the possible excess graphs are the 23 graphs demonstrated in Figure 2.6. We repeat the figure here as a reminder. The excess graphs  $E_1, E_2$ , and  $E_3$  can be achieved from a maximum  $B$ -packing of  $K_{12}$  as we will explain. Label the vertices of  $K_{12}$  with the elements of  $\mathbb{Z}_{12}$ . By Theorem 1.3.2, the leave graph in a maximum  $B$ -packing of  $K_{12}$  has one edge. Let  $R$  be a maximum  $B$ -packing of  $K_{12}$  with the edge  $\{0, 1\}$  as the leave graph. Also consider the sets  $S, U$ , and  $V$  as the sets consisting of the single trees  $(2; 3, 4, 5, 0 - 1)$ ,  $(0; 1, 2, 3, 4 - 5)$ , and  $(0; 2, 3, 4, 1 - 5)$ , respectively. Therefore,  $R \cup S, R \cup U$ , and  $R \cup V$  are minimum  $B$ -coverings of  $K_{12}$  with the excess graphs  $E_1$  (with the edges  $\{0, 2\}, \{2, 3\}, \{2, 4\}$ , and  $\{2, 5\}$ ),  $E_2$  (with the edges  $\{0, 2\}, \{0, 3\}, \{0, 4\}$ , and  $\{4, 5\}$ ), and  $E_3$  (with the edges  $\{0, 2\}, \{0, 3\}, \{0, 4\}$ , and  $\{1, 5\}$ ), respectively.

In order to achieve the excess graph  $E_{12}$ , write  $K_{12} = K_9 \vee K_3$  and label the vertices of  $K_9 \vee K_3$  with a  $(\mathbb{Z}_9, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.2,  $K_9$  has a maximum  $B$ -packing,  $R$ , with one edge, say  $\{7_1, 8_1\}$ , as the leave graph. Consider  $S$  to be the



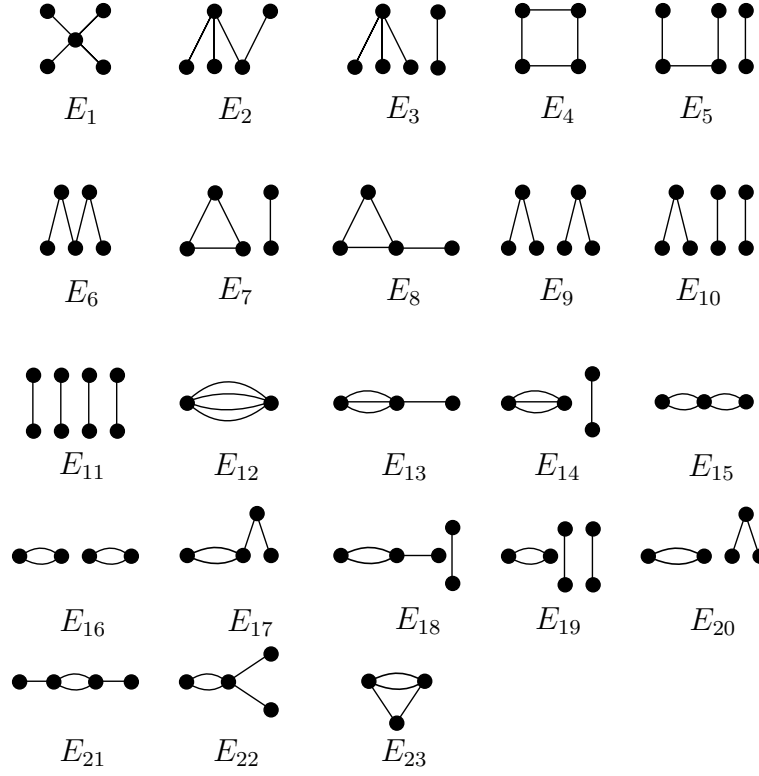


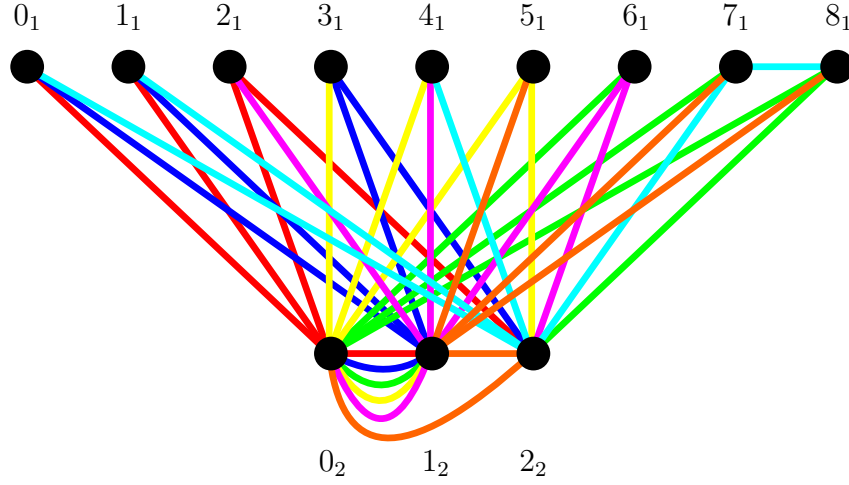
Figure 4.1: All possible 4-edge excess graphs

set consisting of the following trees.

$$\begin{aligned}
 &(0_2; 1_2, 0_1, 1_1, 2_1 - 2_2), (1_2; 0_2, 0_1, 1_1, 3_1 - 2_2), (0_2; 1_2, 3_1, 4_1, 5_1 - 2_2), \\
 &(0_2; 1_2, 6_1, 7_1, 8_1 - 2_2), (1_2; 0_2, 2_1, 4_1, 6_1 - 2_2), (2_2; 0_1, 1_1, 4_1, 7_1 - 8_1), \\
 &(1_2; 5_1, 7_1, 8_1, 2_2 - 0_2)
 \end{aligned}$$

Therefore,  $R \cup S$  forms a minimum  $B$ -covering of  $K_{12}$  with the excess graph  $E_{12}$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used four times. Figure 4.2 demonstrates this covering.

All the remaining excess graphs except for  $E_{11}$  can be obtained by substituting some trees for some others in the covering  $R \cup S$ . Table 4.3 illustrates these substitutions.

Figure 4.2: A  $B$ -covering of  $K_{12}$  with the excess graph  $E_{12}$ 

New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
$(0_2; 2_2, 0_1, 1_1, 2_1 - 3_1)$ $(1_2; 5_1, 7_1, 8_1, 2_2 - 2_1)$ $(0_2; 2_1, 3_1, 4_1, 5_1 - 2_2)$ $(1_2; 3_1, 2_1, 4_1, 6_1 - 2_2)$	$(0_2; 1_2, 0_1, 1_1, 2_1 - 2_2)$ $(1_2; 5_1, 7_1, 8_1, 2_2 - 0_2)$ $(0_2; 1_2, 3_1, 4_1, 5_1 - 2_2)$ $(1_2; 0_2, 2_1, 4_1, 6_1 - 2_2)$	$\{2_1, 3_1\}, \{2_1, 0_2\}, \{3_1, 1_2\}, \{0_2, 1_2\}$	$E_4$
$(0_2; 0_1, 1_1, 2_1, 2_2 - 3_1)$ $(1_2; 5_1, 7_1, 8_1, 2_2 - 2_1)$ $(0_2; 4_1, 6_1, 7_1, 8_1 - 2_2)$ $(1_2; 5_1, 0_1, 1_1, 3_1 - 2_2)$	$(0_2; 1_2, 0_1, 1_1, 2_1 - 2_2)$ $(1_2; 5_1, 7_1, 8_1, 2_2 - 0_2)$ $(0_2; 1_2, 6_1, 7_1, 8_1 - 2_2)$ $(1_2; 0_2, 0_1, 1_1, 3_1 - 2_2)$	$\{3_1, 2_2\}, \{4_1, 0_2\}, \{5_1, 1_2\}, \{0_2, 1_2\}$	$E_5$
$(1_2; 5_1, 2_1, 4_1, 6_1 - 2_2)$ $(0_2; 3_1, 6_1, 7_1, 8_1 - 2_2)$ $(0_2; 0_1, 1_1, 2_1, 2_2 - 3_1)$ $(1_2; 5_1, 7_1, 8_1, 2_2 - 2_1)$	$(1_2; 0_2, 2_1, 4_1, 6_1 - 2_2)$ $(0_2; 1_2, 6_1, 7_1, 8_1 - 2_2)$ $(0_2; 1_2, 0_1, 1_1, 2_1 - 2_2)$ $(1_2; 5_1, 7_1, 8_1, 2_2 - 0_2)$	$\{3_1, 0_2\}, \{3_1, 2_2\}, \{5_1, 1_2\}, \{0_2, 1_2\}$	$E_6$
$(0_2; 3_1, 6_1, 7_1, 8_1 - 2_2)$ $(1_2; 3_1, 2_1, 4_1, 6_1 - 2_2)$ $(0_2; 0_1, 1_1, 2_1, 2_2 - 4_1)$ $(1_2; 5_1, 7_1, 8_1, 2_2 - 2_1)$	$(0_2; 1_2, 6_1, 7_1, 8_1 - 2_2)$ $(1_2; 0_2, 2_1, 4_1, 6_1 - 2_2)$ $(0_2; 1_2, 0_1, 1_1, 2_1 - 2_2)$ $(1_2; 5_1, 7_1, 8_1, 2_2 - 0_2)$	$\{3_1, 0_2\}, \{3_1, 1_2\}, \{4_1, 2_2\}, \{0_2, 1_2\}$	$E_7$
$(0_2; 3_1, 0_1, 1_1, 2_1 - 2_2)$ $(1_2; 3_1, 2_1, 4_1, 6_1 - 2_2)$ $(1_2; 4_1, 0_1, 1_1, 3_1 - 2_2)$	$(0_2; 1_2, 0_1, 1_1, 2_1 - 2_2)$ $(1_2; 0_2, 2_1, 4_1, 6_1 - 2_2)$ $(1_2; 0_2, 0_1, 1_1, 3_1 - 2_2)$	$\{3_1, 0_2\}, \{3_1, 1_2\}, \{4_1, 1_2\}, \{0_2, 1_2\}$	$E_8$
$(0_2; 3_1, 0_1, 1_1, 2_1 - 2_2)$ $(0_2; 4_1, 6_1, 7_1, 8_1 - 2_2)$ $(1_2; 5_1, 2_1, 4_1, 6_1 - 2_2)$ $(1_2; 6_1, 0_1, 1_1, 3_1 - 2_2)$	$(0_2; 1_2, 0_1, 1_1, 2_1 - 2_2)$ $(0_2; 1_2, 6_1, 7_1, 8_1 - 2_2)$ $(1_2; 0_2, 2_1, 4_1, 6_1 - 2_2)$ $(1_2; 0_2, 0_1, 1_1, 3_1 - 2_2)$	$\{3_1, 0_2\}, \{4_1, 0_2\}, \{5_1, 1_2\}, \{6_1, 1_2\}$	$E_9$

New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
(1 <sub>2</sub> ; 5 <sub>1</sub> , 2 <sub>1</sub> , 4 <sub>1</sub> , 6 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 6 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 2 <sub>2</sub> - 3 <sub>1</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> , 2 <sub>2</sub> - 2 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 2 <sub>2</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 2 <sub>1</sub> , 4 <sub>1</sub> , 6 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> , 2 <sub>2</sub> - 0 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 2 <sub>2</sub> )	{5 <sub>1</sub> , 1 <sub>2</sub> }, {6 <sub>1</sub> , 1 <sub>2</sub> }, {4 <sub>1</sub> , 0 <sub>2</sub> }, {3 <sub>1</sub> , 2 <sub>2</sub> }	$E_{10}$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (3 times)	$E_{13}$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 2 <sub>2</sub> - 3 <sub>1</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> , 2 <sub>2</sub> - 2 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> , 2 <sub>2</sub> - 0 <sub>2</sub> )	{3 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (3 times)	$E_{14}$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 3 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 2 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 2 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> } (twice), {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{15}$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 3 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 2 <sub>1</sub> , 4 <sub>1</sub> , 6 <sub>1</sub> - 2 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 2 <sub>1</sub> , 4 <sub>1</sub> , 6 <sub>1</sub> - 2 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> } (twice), {5 <sub>1</sub> , 1 <sub>2</sub> } (twice)	$E_{16}$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 3 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> } (twice), {4 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_{17}$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 2 <sub>2</sub> - 3 <sub>1</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> , 2 <sub>2</sub> - 2 <sub>1</sub> ) (0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> , 2 <sub>2</sub> - 0 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> )	{2 <sub>1</sub> , 0 <sub>2</sub> }, {3 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{18}$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 2 <sub>1</sub> , 4 <sub>1</sub> , 6 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 2 <sub>2</sub> - 6 <sub>1</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> , 2 <sub>2</sub> - 2 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 2 <sub>1</sub> , 4 <sub>1</sub> , 6 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> , 2 <sub>2</sub> - 0 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {5 <sub>1</sub> , 1 <sub>2</sub> } (twice), {6 <sub>1</sub> , 2 <sub>2</sub> }	$E_{19}$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 3 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 5 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 7 <sub>1</sub> , 2 <sub>1</sub> , 4 <sub>1</sub> , 6 <sub>1</sub> - 2 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 2 <sub>1</sub> , 4 <sub>1</sub> , 6 <sub>1</sub> - 2 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> } (twice), {5 <sub>1</sub> , 1 <sub>2</sub> }, {7 <sub>1</sub> , 1 <sub>2</sub> }	$E_{20}$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 2 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {2 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{21}$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {2 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{22}$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 2 <sub>1</sub> , 4 <sub>1</sub> , 6 <sub>1</sub> - 2 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 2 <sub>1</sub> , 4 <sub>1</sub> , 6 <sub>1</sub> - 2 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {3 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{23}$

Table 4.3: Substitutions in the  $B$ -covering  $R \cup S$  to obtain different excess graphs for  $n = 12$

In order to achieve the excess graph  $E_{11}$ , partition and label the vertices of  $K_{12}$

as above and let  $R$  be the same  $B$ -packing of  $K_9$  with the edge  $\{7_1, 8_1\}$  as the leave graph. The set of vertices  $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , and the edges between these two sets, form a complete bipartite graph  $K_{5,3}$ , which has a  $B$ -decomposition,  $U$ , by Lemma 1.3.16. Also let  $V$  be the set consisting of the following trees.

$$\begin{aligned} &(0_2; 1_2, 5_1, 6_1, 7_1 - 2_2), (1_2; 2_2, 6_1, 7_1, 5_1 - 0_2), \\ &(2_2; 0_2, 5_1, 6_1, 7_1 - 8_1), (8_1; 4_1, 0_2, 2_2, 1_2 - 6_1) \end{aligned}$$

Therefore,  $R \cup U \cup V$  forms a minimum  $B$ -covering of  $K_{12}$  with the excess graph  $E_{11}$ . The edges of the excess graph are  $\{4_1, 8_1\}$ ,  $\{5_1, 0_2\}$ ,  $\{6_1, 1_2\}$ , and  $\{7_1, 2_2\}$ .

Case 2.  $T = B, n = 13$

The excess graph has two edges by Theorem 1.3.3. Hence, the possible excess graphs are  $K_2^2$ ,  $P_3$ , and  $2K_2$ . In order to achieve the excess graphs  $P_3$  and  $2K_2$ , label the vertices of  $K_{13}$  with the elements of  $\mathbb{Z}_{13}$ . The leave graph in a maximum  $B$ -packing of  $K_{13}$  has three edges by Theorem 1.3.2 and all possible leave graphs are achievable as we saw in Chapter 3. Let  $R$  be a maximum  $B$ -packing of  $K_{13}$  with the star  $(0; 1, 2, 3)$  as the leave graph. Also let  $S$  be the set consisting of the single tree  $(0; 1, 2, 3, 4 - 5)$  and  $U$  be the set consisting of the single tree  $(0; 5, 1, 2, 3 - 4)$ . Therefore,  $R \cup S$  and  $R \cup U$  are minimum  $B$ -coverings of  $K_{13}$  with the excess graphs  $P_3$  (with the edges  $\{0, 4\}$  and  $\{4, 5\}$ ) and  $2K_2$  (with the edges  $\{0, 5\}$  and  $\{3, 4\}$ ), respectively.

In order to obtain the excess graph  $K_2^2$ , write  $K_{13} = K_{10} \vee K_3$ , label the vertices of  $K_{10} \vee K_3$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.1,  $K_{10}$  has a  $B$ -decomposition,  $R$ . The set of vertices  $\{5_1, 6_1, 7_1, 8_1, 9_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , and the edges between these two sets, form a complete bipartite graph  $K_{5,3}$ , which has a  $B$ -decomposition,  $S$ , by Lemma 1.3.16. Let  $U$  be the set consisting of the following trees.

$$\begin{aligned} &(0_2; 2_2, 0_1, 1_1, 2_1 - 3_1), (3_1; 2_1, 1_2, 2_2, 0_2 - 4_1), \\ &(1_2; 0_1, 1_1, 2_1, 4_1 - 2_2), (2_2; 0_1, 1_1, 2_1, 1_2 - 0_2) \end{aligned}$$

Therefore,  $R \cup S \cup U$  forms a minimum  $B$ -covering of  $K_{13}$  with the excess graph  $K_2^2$ . The edges of the excess graph are the edges  $\{2_1, 3_1\}$  used twice.

Case 3.  $T = B, n = 14$

The excess graph has four edges by Theorem 1.3.3. Hence, the possible excess graphs are those in Figure 4.1. In order to obtain the excess graphs  $E_1, E_2$ , and  $E_3$ , label the vertices of  $K_{14}$  with the elements of  $\mathbb{Z}_{14}$ . By Theorem 1.3.2, the leave graph in a maximum  $B$ -packing of  $K_{14}$  has one edge. Let  $R$  be a maximum  $B$ -packing of  $K_{14}$  with the edge  $\{0, 1\}$  as the leave graph. Also let  $S$  be the set consisting of the single tree  $(2; 3, 4, 5, 1 - 0)$ ,  $U$  be the set consisting of the single tree  $(1; 0, 2, 3, 4 - 5)$ , and  $V$  be the set consisting of the single tree  $(1; 2, 3, 4, 0 - 5)$ . Therefore,  $R \cup S$ ,  $R \cup U$ , and  $R \cup V$  are minimum  $B$ -coverings of  $K_{14}$  with the excess graphs  $E_1$  (with the edges  $\{1, 2\}, \{2, 3\}, \{2, 4\}$ , and  $\{2, 5\}$ ),  $E_2$  (with the edges  $\{1, 2\}, \{1, 3\}, \{1, 4\}$ , and  $\{4, 5\}$ ), and  $E_3$  (with the edges  $\{1, 2\}, \{1, 3\}, \{1, 4\}$ , and  $\{0, 5\}$ ), respectively.

In order to achieve the excess graph  $E_{12}$ , write  $K_{14} = K_9 \vee K_5$ , label the vertices of  $K_9 \vee K_5$  with a  $(\mathbb{Z}_9, \mathbb{Z}_5)$ -labeling. By Theorem 1.3.2, the leave graph in a maximum  $B$ -packing of  $K_9$  has one edge. Let  $R$  be a maximum  $B$ -packing of  $K_9$  with the edge  $\{7_1, 8_1\}$  as the leave graph. Also let  $S$  be the set consisting of the following trees.

$$\begin{aligned} & (0_2; 1_2, 0_1, 1_1, 2_1 - 2_2), (1_2; 0_2, 0_1, 1_1, 2_1 - 3_2), (0_2; 1_2, 3_1, 4_1, 5_1 - 2_2), \\ & (1_2; 0_2, 3_1, 4_1, 5_1 - 3_2), (0_2; 1_2, 6_1, 7_1, 8_1 - 3_2), (1_2; 6_1, 2_2, 3_2, 4_2 - 0_2), \\ & (2_2; 0_1, 1_1, 3_1, 4_1 - 3_2), (3_2; 0_1, 1_1, 3_1, 6_1 - 4_2), (4_2; 0_1, 1_1, 2_1, 3_2 - 2_2), \\ & (4_2; 3_1, 4_1, 5_1, 8_1 - 7_1), (7_1; 1_2, 2_2, 4_2, 3_2 - 0_2), (2_2; 0_2, 4_2, 6_1, 8_1 - 1_2) \end{aligned}$$

Therefore,  $R \cup S$  is a minimum  $B$ -covering of  $K_{14}$  with the excess graph  $E_{12}$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used four times.

All the remaining excess graphs will be achieved by substituting some trees with others in the covering  $R \cup S$ . Table 4.4 illustrates these substitutions.

Case 4.  $T = C, n = 12$

The excess graph has four edges by Theorem 1.3.3. Hence, the possible excess graphs are the ones shown in Figure 4.1. In order to achieve the excess graphs  $E_2, E_3, E_6$ , and  $E_9$ , label the vertices of  $K_{12}$  with the elements of  $\mathbb{Z}_{12}$ . By Theorem 1.3.2, the leave graph in a maximum  $C$ -packing of  $K_{12}$  has one edge. Let  $R$  be a maximum  $C$ -packing of  $K_{12}$  with the edge  $\{0, 1\}$  as the leave graph. Also let  $S$  be the set consisting of the single tree  $(3; 4, 5, 2 - 1 - 0)$ ,  $U$  be the set consisting of the single tree  $(5; 3, 4, 1 - 0 - 2)$ ,  $V$  be the set consisting of the single tree  $(1; 0, 2, 3 - 4 - 5)$ , and

New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 2 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 3 <sub>2</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 3 <sub>2</sub> ) (1 <sub>2</sub> ; 2 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 3 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 3 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 3 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 3 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {3 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 2 <sub>2</sub> }, {1 <sub>2</sub> , 2 <sub>2</sub> }	$E_4$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 2 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 3 <sub>2</sub> ) (0 <sub>2</sub> ; 4 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 6 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> - 8 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 3 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 6 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> - 0 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {1 <sub>2</sub> , 0 <sub>2</sub> }, {8 <sub>1</sub> , 4 <sub>2</sub> }, {1 <sub>2</sub> , 2 <sub>2</sub> }	$E_5$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 2 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 3 <sub>2</sub> ) (0 <sub>2</sub> ; 4 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 6 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> - 3 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 3 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 6 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> - 0 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {3 <sub>1</sub> , 4 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }, {1 <sub>2</sub> , 2 <sub>2</sub> }	$E_6$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 3 <sub>2</sub> ) (0 <sub>2</sub> ; 4 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 6 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> - 8 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 3 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 6 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> - 0 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {3 <sub>1</sub> , 1 <sub>2</sub> }, {8 <sub>1</sub> , 4 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_7$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 3 <sub>2</sub> ) (0 <sub>2</sub> ; 4 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 3 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {3 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 4 <sub>2</sub> }	$E_8$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 6 <sub>1</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 3 <sub>2</sub> ) (1 <sub>2</sub> ; 7 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 3 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 3 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 3 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {2 <sub>1</sub> , 0 <sub>2</sub> }, {6 <sub>1</sub> , 1 <sub>2</sub> }, {7 <sub>1</sub> , 1 <sub>2</sub> }	$E_9$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 2 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 3 <sub>2</sub> ) (2 <sub>2</sub> ; 6 <sub>1</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> , 8 <sub>1</sub> - 1 <sub>2</sub> ) (1 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 3 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 3 <sub>2</sub> ) (2 <sub>2</sub> ; 6 <sub>1</sub> , 0 <sub>2</sub> , 4 <sub>2</sub> , 8 <sub>1</sub> - 1 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 3 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {2 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 1 <sub>2</sub> }, {2 <sub>2</sub> , 3 <sub>2</sub> }	$E_{10}$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 3 <sub>2</sub> ) (0 <sub>2</sub> ; 4 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 6 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> - 8 <sub>1</sub> ) (0 <sub>2</sub> ; 2 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 3 <sub>2</sub> ) (2 <sub>2</sub> ; 6 <sub>1</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> , 8 <sub>1</sub> - 1 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 3 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 6 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> - 0 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 3 <sub>2</sub> ) (2 <sub>2</sub> ; 6 <sub>1</sub> , 0 <sub>2</sub> , 4 <sub>2</sub> , 8 <sub>1</sub> - 1 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 1 <sub>2</sub> }, {8 <sub>1</sub> , 4 <sub>2</sub> }, {2 <sub>2</sub> , 3 <sub>2</sub> }	$E_{11}$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }(3 times)	$E_{13}$
(0 <sub>2</sub> ; 4 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 6 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> - 8 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 6 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 4 <sub>2</sub> - 0 <sub>2</sub> )	{8 <sub>1</sub> , 4 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }(3 times)	$E_{14}$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 3 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 3 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 3 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }(twice), {0 <sub>2</sub> , 1 <sub>2</sub> }(twice)	$E_{15}$
(0 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 3 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 3 <sub>2</sub> ) (1 <sub>2</sub> ; 6 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 3 <sub>2</sub> ) (1 <sub>2</sub> ; 6 <sub>1</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 3 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> , 8 <sub>1</sub> - 3 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> - 3 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 3 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 3 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }(twice), {6 <sub>1</sub> , 1 <sub>2</sub> }(twice)	$E_{16}$

New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
$(0_2; 3_1, 0_1, 1_1, 2_1 - 2_2)$ $(0_2; 4_2, 3_1, 4_1, 5_1 - 2_2)$ $(1_2; 6_1, 2_2, 3_2, 4_2 - 3_1)$	$(0_2; 1_2, 0_1, 1_1, 2_1 - 2_2)$ $(0_2; 1_2, 3_1, 4_1, 5_1 - 2_2)$ $(1_2; 6_1, 2_2, 3_2, 4_2 - 0_2)$	$\{3_1, 0_2\}, \{3_1, 4_2\}, \{0_2, 1_2\}$ (twice)	$E_{17}$
$(0_2; 3_1, 0_1, 1_1, 2_1 - 2_2)$ $(0_2; 4_2, 6_1, 7_1, 8_1 - 3_2)$ $(1_2; 6_1, 2_2, 3_2, 4_2 - 8_1)$	$(0_2; 1_2, 0_1, 1_1, 2_1 - 2_2)$ $(0_2; 1_2, 6_1, 7_1, 8_1 - 3_2)$ $(1_2; 6_1, 2_2, 3_2, 4_2 - 0_2)$	$\{3_1, 0_2\}, \{8_1, 4_2\}, \{0_2, 1_2\}$ (twice)	$E_{18}$
$(0_2; 3_2, 0_1, 1_1, 2_1 - 2_2)$ $(7_1; 1_2, 2_2, 4_2, 3_2 - 3_1)$ $(0_2; 4_2, 6_1, 7_1, 8_1 - 3_2)$ $(1_2; 6_1, 2_2, 3_2, 4_2 - 4_1)$	$(0_2; 1_2, 0_1, 1_1, 2_1 - 2_2)$ $(7_1; 1_2, 2_2, 4_2, 3_2 - 0_2)$ $(0_2; 1_2, 6_1, 7_1, 8_1 - 3_2)$ $(1_2; 6_1, 2_2, 3_2, 4_2 - 0_2)$	$\{3_1, 3_2\}, \{4_1, 4_2\}, \{0_2, 1_2\}$ (twice)	$E_{19}$
$(0_2; 3_2, 0_1, 1_1, 2_1 - 2_2)$ $(7_1; 1_2, 2_2, 4_2, 3_2 - 3_1)$ $(0_2; 4_2, 6_1, 7_1, 8_1 - 3_2)$ $(1_2; 6_1, 2_2, 3_2, 4_2 - 3_1)$	$(0_2; 1_2, 0_1, 1_1, 2_1 - 2_2)$ $(7_1; 1_2, 2_2, 4_2, 3_2 - 0_2)$ $(0_2; 1_2, 6_1, 7_1, 8_1 - 3_2)$ $(1_2; 6_1, 2_2, 3_2, 4_2 - 0_2)$	$\{3_1, 3_2\}, \{3_1, 4_2\}, \{0_2, 1_2\}$ (twice)	$E_{20}$
$(0_2; 3_1, 0_1, 1_1, 2_1 - 2_2)$ $(1_2; 4_1, 0_1, 1_1, 2_1 - 3_2)$	$(0_2; 1_2, 0_1, 1_1, 2_1 - 2_2)$ $(1_2; 0_2, 0_1, 1_1, 2_1 - 3_2)$	$\{3_1, 0_2\}, \{4_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$E_{21}$
$(0_2; 3_1, 0_1, 1_1, 2_1 - 2_2)$ $(0_2; 4_1, 6_1, 7_1, 8_1 - 3_2)$	$(0_2; 1_2, 0_1, 1_1, 2_1 - 2_2)$ $(0_2; 1_2, 6_1, 7_1, 8_1 - 3_2)$	$\{3_1, 0_2\}, \{4_1, 0_2\}, \{0_2, 1_2\}$ (twice)	$E_{22}$
$(0_2; 3_1, 0_1, 1_1, 2_1 - 2_2)$ $(1_2; 3_1, 0_1, 1_1, 2_1 - 3_2)$	$(0_2; 1_2, 0_1, 1_1, 2_1 - 2_2)$ $(1_2; 0_2, 0_1, 1_1, 2_1 - 3_2)$	$\{3_1, 0_2\}, \{4_1, 0_2\}, \{0_2, 1_2\}$ (twice)	$E_{23}$

Table 4.4: Substitutions in the  $B$ -covering  $R \cup S$  to obtain different excess graphs for  $n = 14$

$W$  be the set consisting of the single tree  $(1; 2, 3, 0 - 4 - 5)$ . Then  $R \cup S, R \cup U, R \cup V$ , and  $R \cup W$  are minimum  $C$ -coverings of  $K_{12}$  with the excess graphs  $E_2$  (with the edges  $\{1, 2\}, \{2, 3\}, \{3, 4\}$ , and  $\{3, 5\}$ ),  $E_3$  (with the edges  $\{0, 2\}, \{1, 5\}, \{3, 5\}$ , and  $\{4, 5\}$ ),  $E_6$  (with the edges  $\{1, 2\}, \{1, 3\}, \{3, 4\}$ , and  $\{4, 5\}$ ), and  $E_9$  (with the edges  $\{0, 4\}, \{0, 5\}, \{1, 2\}$ , and  $\{1, 3\}$ ), respectively.

In order to obtain the excess graph  $E_{12}$ , write  $K_{12} = K_9 \vee K_3$  and label the vertices of  $K_9 \vee K_3$  with a  $(\mathbb{Z}_9, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.2, the leave graph in a maximum  $C$ -packing of  $K_9$  has one edge. Let  $R$  be a maximum  $C$ -packing of  $K_9$  with the edge  $\{7_1, 8_1\}$  as the leave graph. Also let  $S$  be the set consisting of the following trees.

$$\begin{aligned}
&(0_2; 0_1, 1_1, 1_2 - 2_1 - 2_2), (0_2; 2_1, 3_1, 1_2 - 4_1 - 2_2), (0_2; 1_2, 4_1, 5_1 - 2_2 - 6_1), \\
&(0_2; 1_2, 6_1, 7_1 - 8_1 - 2_2), (1_2; 0_2, 0_1, 1_1 - 2_2 - 3_1), (1_2; 3_1, 5_1, 2_2 - 0_2 - 8_1), \\
&(1_2; 6_1, 8_1, 7_1 - 2_2 - 0_1)
\end{aligned}$$

Therefore,  $R \cup S$  is a minimum  $C$ -covering of  $K_{12}$  with the excess graph  $E_{12}$ . The edges of the excess graph are  $\{0_2, 1_2\}$ .

All the remaining excess graphs will be achieved by substituting some trees for

others in the covering  $R \cup S$ . Table 4.5 illustrates these substitutions.

Case 5.  $T = C, n = 13$

By Theorem 1.3.3, the excess graph has two edges. Hence the possible excess graphs are  $K_2^2, P_3$ , and  $2K_2$ . The excess graphs  $P_3$  and  $2K_2$  will be obtained from a maximum  $C$ -packing of  $K_{13}$ . Label the vertices of  $K_{13}$  with the elements of  $\mathbb{Z}_{13}$ . By Theorem 1.3.2, the leave graph in a maximum  $C$ -packing of  $K_{13}$  has three edges. Moreover, all simple graphs with three edges can be achieved as the leave graph in maximum  $C$ -packings of  $K_{13}$  as illustrated in Chapter 3. Let  $R$  be a maximum  $C$ -packing of  $K_{13}$  with the edges  $\{0, 1\}, \{0, 2\}$ , and  $\{3, 4\}$  as the edges of the leave graph. Also let  $S$  be the set consisting of the single tree  $(4; 5, 3, 2 - 0 - 1)$  and  $U$  be the set consisting of the single tree  $(0; 1, 2, 3 - 4 - 5)$ . Therefore,  $R \cup S$  and  $R \cup U$  are minimum  $C$ -coverings of  $K_{13}$  with the excess graphs  $P_3$  (with edges  $\{2, 4\}$  and  $\{2, 5\}$ ) and  $2K_2$  (with the edges  $\{0, 3\}$  and  $\{4, 5\}$ ), respectively.

In order to achieve the excess graph  $K_2^2$ , write  $K_{13} = K_{10} \vee K_3$  and label the vertices of  $K_{10} \vee K_3$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.1,  $K_{10}$  has a  $C$ -decomposition,  $R$ . Furthermore, the set of vertices  $\{5_1, 6_1, 7_1, 8_1, 9_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , and the edges between these two sets, form a graph  $K_{5,3}$ , which has a  $C$ -decomposition,  $S$ , by Lemma 1.3.16. Now, let  $U$  be the set consisting of the following trees.

$$(0_2; 1_2, 0_1, 1_1 - 2_2 - 2_1), (2_2; 3_1, 4_1, 1_2 - 0_2 - 2_1), \\ (1_2; 2_1, 3_1, 0_2 - 2_2 - 0_1), (1_2; 0_1, 1_1, 4_1 - 0_2 - 3_1)$$

Therefore,  $R \cup S \cup U$  forms a minimum  $C$ -covering of  $K_{13}$  with the excess graph  $K_2^2$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used twice.

Case 6.  $T = C, n = 14$

The excess graph has four edges by Theorem 1.3.3. Hence, the possible excess graphs are those illustrated in Figure 4.1. Since a maximum  $C$ -packing of  $K_{14}$  has one edge by Theorem 1.3.2, the excess graphs  $E_2, E_3, E_6$ , and  $E_9$  can be achieved as explained in Case 4.

In order to obtain the excess graph  $E_{12}$ , write  $K_{14} = K_{11} \vee K_3$  and label the vertices of  $K_{11} \vee K_3$  with a  $(\mathbb{Z}_{11}, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.1,  $K_{11}$  has a  $C$ -decomposition,  $R$ . Moreover, the set of vertices  $\{6_1, 7_1, 8_1, 9_1, 10_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , and the edges between these two sets, form a graph  $K_{5,3}$ , which has a  $C$ -decomposition,



New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
(0 <sub>2</sub> ; 0 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> - 6 <sub>1</sub> ) (2 <sub>2</sub> ; 0 <sub>2</sub> , 3 <sub>1</sub> , 1 <sub>1</sub> - 1 <sub>2</sub> - 0 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> - 6 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> )	{0 <sub>1</sub> , 0 <sub>2</sub> }, {1 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 2 <sub>2</sub> }	$E_1$
(2 <sub>2</sub> ; 0 <sub>2</sub> , 3 <sub>1</sub> , 1 <sub>1</sub> - 1 <sub>2</sub> - 0 <sub>1</sub> ) (8 <sub>1</sub> ; 1 <sub>2</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> - 0 <sub>2</sub> - 6 <sub>1</sub> ) (2 <sub>2</sub> ; 8 <sub>1</sub> , 6 <sub>1</sub> , 5 <sub>1</sub> - 0 <sub>2</sub> - 4 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> - 6 <sub>1</sub> )	{8 <sub>1</sub> , 1 <sub>2</sub> }, {8 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 2 <sub>2</sub> }	$E_4$
(0 <sub>2</sub> ; 2 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (2 <sub>2</sub> ; 8 <sub>1</sub> , 6 <sub>1</sub> , 5 <sub>1</sub> - 0 <sub>2</sub> - 4 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> - 6 <sub>1</sub> )	{2 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 1 <sub>2</sub> }, {8 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_5$
(0 <sub>2</sub> ; 2 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 2 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (2 <sub>2</sub> ; 8 <sub>1</sub> , 6 <sub>1</sub> , 5 <sub>1</sub> - 0 <sub>2</sub> - 4 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> - 6 <sub>1</sub> )	{2 <sub>1</sub> , 0 <sub>2</sub> }, {2 <sub>1</sub> , 1 <sub>2</sub> }, {8 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_7$
(0 <sub>2</sub> ; 2 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 2 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (2 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 5 <sub>1</sub> - 0 <sub>2</sub> - 4 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> - 6 <sub>1</sub> )	{2 <sub>1</sub> , 0 <sub>2</sub> }, {2 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }, {1 <sub>2</sub> , 2 <sub>2</sub> }	$E_8$
(1 <sub>2</sub> ; 2 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (8 <sub>1</sub> ; 3 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> - 0 <sub>2</sub> - 6 <sub>1</sub> ) (2 <sub>2</sub> ; 0 <sub>1</sub> , 6 <sub>1</sub> , 5 <sub>1</sub> - 0 <sub>2</sub> - 4 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> - 6 <sub>1</sub> )	{3 <sub>1</sub> , 8 <sub>1</sub> }, {0 <sub>1</sub> , 2 <sub>2</sub> }, {2 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_{10}$
(2 <sub>1</sub> ; 1 <sub>2</sub> , 2 <sub>2</sub> , 1 <sub>1</sub> - 0 <sub>2</sub> - 0 <sub>1</sub> ) (4 <sub>1</sub> ; 1 <sub>2</sub> , 2 <sub>2</sub> , 3 <sub>1</sub> - 0 <sub>2</sub> - 2 <sub>1</sub> ) (2 <sub>2</sub> ; 8 <sub>1</sub> , 6 <sub>1</sub> , 5 <sub>1</sub> - 0 <sub>2</sub> - 4 <sub>1</sub> )	(0 <sub>2</sub> ; 0 <sub>1</sub> , 1 <sub>1</sub> , 1 <sub>2</sub> - 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 1 <sub>2</sub> - 4 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> - 6 <sub>1</sub> )	{1 <sub>1</sub> , 2 <sub>1</sub> }, {3 <sub>1</sub> , 4 <sub>1</sub> }, {8 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_{11}$
(0 <sub>2</sub> ; 5 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> )	{5 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (3 times)	$E_{13}$
(2 <sub>2</sub> ; 3 <sub>1</sub> , 4 <sub>1</sub> , 1 <sub>1</sub> - 1 <sub>2</sub> - 0 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> )	{4 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (3 times)	$E_{14}$
(1 <sub>2</sub> ; 8 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (8 <sub>1</sub> ; 1 <sub>2</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> - 0 <sub>2</sub> - 6 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> )	{8 <sub>1</sub> , 1 <sub>2</sub> } (twice), {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{15}$
(2 <sub>2</sub> ; 8 <sub>1</sub> , 6 <sub>1</sub> , 5 <sub>1</sub> - 0 <sub>2</sub> - 4 <sub>1</sub> ) (2 <sub>2</sub> ; 8 <sub>1</sub> , 3 <sub>1</sub> , 1 <sub>1</sub> - 1 <sub>2</sub> - 0 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> - 6 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> )	{8 <sub>1</sub> , 2 <sub>2</sub> } (twice), {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{16}$
(2 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 5 <sub>1</sub> - 0 <sub>2</sub> - 4 <sub>1</sub> ) (2 <sub>2</sub> ; 3 <sub>1</sub> , 8 <sub>1</sub> , 1 <sub>1</sub> - 1 <sub>2</sub> - 0 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> - 6 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> )	{8 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice), {1 <sub>2</sub> , 2 <sub>2</sub> }	$E_{17}$
(2 <sub>2</sub> ; 3 <sub>1</sub> , 4 <sub>1</sub> , 1 <sub>1</sub> - 1 <sub>2</sub> - 0 <sub>1</sub> ) (0 <sub>2</sub> ; 0 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> )	{0 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{18}$
(2 <sub>2</sub> ; 3 <sub>1</sub> , 4 <sub>1</sub> , 1 <sub>1</sub> - 1 <sub>2</sub> - 0 <sub>1</sub> ) (8 <sub>1</sub> ; 5 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> - 0 <sub>2</sub> - 6 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> )	{5 <sub>1</sub> , 8 <sub>1</sub> }, {4 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{19}$
(2 <sub>2</sub> ; 3 <sub>1</sub> , 4 <sub>1</sub> , 1 <sub>1</sub> - 1 <sub>2</sub> - 0 <sub>1</sub> ) (8 <sub>1</sub> ; 4 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> - 0 <sub>2</sub> - 6 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> )	{4 <sub>1</sub> , 8 <sub>1</sub> }, {4 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{20}$
(2 <sub>2</sub> ; 0 <sub>2</sub> , 3 <sub>1</sub> , 1 <sub>1</sub> - 1 <sub>2</sub> - 0 <sub>1</sub> ) (8 <sub>1</sub> ; 1 <sub>2</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> - 0 <sub>2</sub> - 6 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> )	{8 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice), {0 <sub>2</sub> , 2 <sub>2</sub> }	$E_{21}$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> ) (2 <sub>2</sub> ; 0 <sub>2</sub> , 3 <sub>1</sub> , 1 <sub>1</sub> - 1 <sub>2</sub> - 0 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 6 <sub>1</sub> , 7 <sub>1</sub> - 8 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> )	{0 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice), {0 <sub>2</sub> , 2 <sub>2</sub> }	$E_{22}$
(0 <sub>2</sub> ; 2 <sub>1</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> - 6 <sub>1</sub> ) (1 <sub>2</sub> ; 2 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> )	(0 <sub>2</sub> ; 1 <sub>2</sub> , 4 <sub>1</sub> , 5 <sub>1</sub> - 2 <sub>2</sub> - 6 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> )	{2 <sub>1</sub> , 0 <sub>2</sub> }, {2 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{23}$

Table 4.5: Substitutions in the  $C$ -covering  $R \cup S$  to obtain different excess graphs for  $n = 12$

$S$ , by Lemma 1.3.16. Also let  $U$  be the set consisting of the following trees.

$$\begin{aligned} & (0_2; 0_1, 1_1, 1_2 - 2_1 - 2_2), (0_2; 2_1, 3_1, 1_2 - 4_1 - 2_2), (1_2; 0_2, 0_1, 1_1 - 2_2 - 5_1) \\ & (1_2; 3_1, 5_1, 0_2 - 2_2 - 0_1), (0_2; 4_1, 5_1, 1_2 - 2_2 - 3_1) \end{aligned}$$

Therefore,  $R \cup S \cup U$  forms a minimum  $C$ -covering of  $K_{14}$  with the excess graph  $E_{12}$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used four times.

In order to achieve the remaining excess graphs, we substitute some trees for others in the covering  $R \cup S \cup U$ . Table 4.6 illustrates these substitutions.

Case 7.  $T = D, n = 12$

The excess graph has four edges by Theorem 1.3.3. Hence, the possible excess graphs are those illustrated in Figure 4.1. In order to achieve the excess graphs  $E_2, E_5$ , and  $E_6$ , label the vertices of  $K_{12}$  with the elements of  $\mathbb{Z}_{12}$ . By Theorem 1.3.2, the leave graph in a maximum  $D$ -packing of  $K_{12}$  has one edge. Let  $R$  be a maximum  $D$ -packing of  $K_{12}$  with the single edge  $\{0, 1\}$  as the leave graph. Also let  $S$  be the set consisting of the single tree  $(3; 2, 1, 4 - 0, 5)$ ,  $U$  be the set consisting of the single tree  $(0; 3, 1, 4 - 2, 5)$ , and  $V$  be the set consisting of the single tree  $(1; 0, 2, 3 - 4, 5)$ . Therefore,  $R \cup S, R \cup U$ , and  $R \cup V$  are minimum  $D$ -coverings of  $K_{12}$  with the excess graphs  $E_2$  (with the edges  $\{1, 3\}, \{2, 3\}, \{3, 4\}$ , and  $\{4, 5\}$ ),  $E_5$  (with the edges  $\{0, 3\}, \{0, 4\}, \{1, 2\}$ , and  $\{4, 5\}$ ), and  $E_6$  (with the edges  $\{1, 2\}, \{1, 3\}, \{2, 4\}$ , and  $\{3, 5\}$ ).

In order to achieve the excess graph  $E_{12}$ , write  $K_{12} = K_9 \vee K_3$  and label the vertices of  $K_9 \vee K_3$  with a  $(\mathbb{Z}_9, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.2, the leave graph in a maximum  $D$ -packing of  $K_9$  has one edge. Let  $R$  be a maximum  $D$ -packing of  $K_9$  with the edge  $\{7_1, 8_1\}$  as the leave graph. Also let  $S$  be the set consisting of the following trees.

$$\begin{aligned} & (0_2; 7_1, 1_1, 1_2 - 2_2, 2_1), (0_2; 2_1, 3_1, 1_2 - 2_2, 0_1), (1_2; 3_1, 5_1, 0_2 - 2_2, 4_1), \\ & (1_2; 1_1, 4_1, 0_2 - 2_2, 5_1), (1_2; 7_1, 6_1, 0_2 - 2_2, 8_1), (2_2; 1_2, 7_1, 0_2 - 8_1, 6_1), \\ & (2_2; 2_1, 0_1, 8_1 - 0_2, 1_2) \end{aligned}$$

Therefore,  $R \cup S$  forms a minimum  $D$ -covering of  $K_{12}$  with the excess graph  $E_{12}$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used four times.

New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
(4 <sub>1</sub> ; 1 <sub>2</sub> , 2 <sub>2</sub> , 3 <sub>1</sub> - 0 <sub>2</sub> - 2 <sub>1</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (2 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 3 <sub>1</sub> - 1 <sub>2</sub> - 5 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 5 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> - 1 <sub>2</sub> )	(0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 1 <sub>2</sub> - 4 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 5 <sub>1</sub> , 0 <sub>2</sub> - 2 <sub>2</sub> - 0 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 5 <sub>1</sub> , 1 <sub>2</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> )	{3 <sub>1</sub> , 4 <sub>1</sub> }, {3 <sub>1</sub> , 0 <sub>2</sub> }, {3 <sub>1</sub> , 1 <sub>2</sub> }, {3 <sub>1</sub> , 2 <sub>2</sub> }	$E_1$
(4 <sub>1</sub> ; 1 <sub>2</sub> , 2 <sub>2</sub> , 3 <sub>1</sub> - 0 <sub>2</sub> - 2 <sub>1</sub> ) (1 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 5 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> - 1 <sub>2</sub> )	(0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 1 <sub>2</sub> - 4 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 5 <sub>1</sub> , 1 <sub>2</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> )	{3 <sub>1</sub> , 4 <sub>1</sub> }, {3 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_4$
(1 <sub>2</sub> ; 4 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 5 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> - 1 <sub>2</sub> ) (2 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 5 <sub>1</sub> - 1 <sub>2</sub> - 3 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 5 <sub>1</sub> , 1 <sub>2</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 5 <sub>1</sub> , 0 <sub>2</sub> - 2 <sub>2</sub> - 0 <sub>1</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {4 <sub>1</sub> , 1 <sub>2</sub> }, {5 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_5$
(1 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 5 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> - 1 <sub>2</sub> ) (2 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 5 <sub>1</sub> - 1 <sub>2</sub> - 3 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 5 <sub>1</sub> , 1 <sub>2</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 5 <sub>1</sub> , 0 <sub>2</sub> - 2 <sub>2</sub> - 0 <sub>1</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {3 <sub>1</sub> , 1 <sub>2</sub> }, {5 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_7$
(1 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 5 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> - 1 <sub>2</sub> ) (0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 2 <sub>2</sub> - 4 <sub>1</sub> - 1 <sub>2</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (0 <sub>2</sub> ; 4 <sub>1</sub> , 5 <sub>1</sub> , 1 <sub>2</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 1 <sub>2</sub> - 4 <sub>1</sub> - 2 <sub>2</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {3 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 2 <sub>2</sub> }	$E_8$
(1 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (2 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 5 <sub>1</sub> - 1 <sub>2</sub> - 3 <sub>1</sub> ) (2 <sub>1</sub> ; 1 <sub>2</sub> , 2 <sub>2</sub> , 1 <sub>1</sub> - 0 <sub>2</sub> - 0 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 5 <sub>1</sub> , 0 <sub>2</sub> - 2 <sub>2</sub> - 0 <sub>1</sub> ) (0 <sub>2</sub> ; 0 <sub>1</sub> , 1 <sub>1</sub> , 1 <sub>2</sub> - 2 <sub>1</sub> - 2 <sub>2</sub> )	{1 <sub>1</sub> , 2 <sub>1</sub> }, {3 <sub>1</sub> , 1 <sub>2</sub> }, {5 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_{10}$
(2 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 5 <sub>1</sub> - 1 <sub>2</sub> - 3 <sub>1</sub> ) (2 <sub>1</sub> ; 1 <sub>2</sub> , 2 <sub>2</sub> , 1 <sub>1</sub> - 0 <sub>2</sub> - 0 <sub>1</sub> ) (4 <sub>1</sub> ; 1 <sub>2</sub> , 2 <sub>2</sub> , 3 <sub>1</sub> - 0 <sub>2</sub> - 2 <sub>1</sub> )	(1 <sub>2</sub> ; 3 <sub>1</sub> , 5 <sub>1</sub> , 0 <sub>2</sub> - 2 <sub>2</sub> - 0 <sub>1</sub> ) (0 <sub>2</sub> ; 0 <sub>1</sub> , 1 <sub>1</sub> , 1 <sub>2</sub> - 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 1 <sub>2</sub> - 4 <sub>1</sub> - 2 <sub>2</sub> )	{1 <sub>1</sub> , 2 <sub>1</sub> }, {3 <sub>1</sub> , 4 <sub>1</sub> }, {5 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_{11}$
(1 <sub>2</sub> ; 2 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> )	{2 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (3 times)	$E_{13}$
(2 <sub>2</sub> ; 4 <sub>1</sub> , 5 <sub>1</sub> , 1 <sub>1</sub> - 1 <sub>2</sub> - 0 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> )	{4 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (3 times)	$E_{14}$
(0 <sub>2</sub> ; 0 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>2</sub> - 2 <sub>1</sub> - 1 <sub>2</sub> ) (0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 2 <sub>2</sub> - 4 <sub>1</sub> - 1 <sub>2</sub> )	(0 <sub>2</sub> ; 0 <sub>1</sub> , 1 <sub>1</sub> , 1 <sub>2</sub> - 2 <sub>1</sub> - 2 <sub>2</sub> ) (0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 1 <sub>2</sub> - 4 <sub>1</sub> - 2 <sub>2</sub> )	{0 <sub>2</sub> , 1 <sub>2</sub> } (twice), {0 <sub>2</sub> , 2 <sub>2</sub> } (twice)	$E_{15}$
(2 <sub>2</sub> ; 3 <sub>1</sub> , 5 <sub>1</sub> , 1 <sub>1</sub> - 1 <sub>2</sub> - 0 <sub>1</sub> ) (2 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 3 <sub>1</sub> - 1 <sub>2</sub> - 5 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 5 <sub>1</sub> , 0 <sub>2</sub> - 2 <sub>2</sub> - 0 <sub>1</sub> )	{3 <sub>1</sub> , 2 <sub>2</sub> } (twice), {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{16}$
(1 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (2 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 3 <sub>1</sub> - 1 <sub>2</sub> - 5 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 5 <sub>1</sub> , 0 <sub>2</sub> - 2 <sub>2</sub> - 0 <sub>1</sub> )	{3 <sub>1</sub> , 1 <sub>2</sub> }, {3 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{17}$
(1 <sub>2</sub> ; 2 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (2 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 3 <sub>1</sub> - 1 <sub>2</sub> - 5 <sub>1</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 5 <sub>1</sub> , 0 <sub>2</sub> - 2 <sub>2</sub> - 0 <sub>1</sub> )	{2 <sub>1</sub> , 1 <sub>2</sub> }, {3 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{18}$
(2 <sub>1</sub> ; 1 <sub>2</sub> , 2 <sub>2</sub> , 1 <sub>1</sub> - 0 <sub>2</sub> - 0 <sub>1</sub> ) (2 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 3 <sub>1</sub> - 1 <sub>2</sub> - 5 <sub>1</sub> )	(0 <sub>2</sub> ; 0 <sub>1</sub> , 1 <sub>1</sub> , 1 <sub>2</sub> - 2 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 5 <sub>1</sub> , 0 <sub>2</sub> - 2 <sub>2</sub> - 0 <sub>1</sub> )	{1 <sub>1</sub> , 2 <sub>1</sub> }, {3 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{19}$
(4 <sub>1</sub> ; 1 <sub>2</sub> , 2 <sub>2</sub> , 3 <sub>1</sub> - 0 <sub>2</sub> - 2 <sub>1</sub> ) (2 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 3 <sub>1</sub> - 1 <sub>2</sub> - 5 <sub>1</sub> )	(0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 1 <sub>2</sub> - 4 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 5 <sub>1</sub> , 0 <sub>2</sub> - 2 <sub>2</sub> - 0 <sub>1</sub> )	{3 <sub>1</sub> , 4 <sub>1</sub> }, {3 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{20}$
(0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 2 <sub>2</sub> - 4 <sub>1</sub> - 1 <sub>2</sub> ) (1 <sub>2</sub> ; 2 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> )	(0 <sub>2</sub> ; 2 <sub>1</sub> , 3 <sub>1</sub> , 1 <sub>2</sub> - 4 <sub>1</sub> - 2 <sub>2</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> )	{2 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice), {0 <sub>2</sub> , 2 <sub>2</sub> }	$E_{21}$
(1 <sub>2</sub> ; 2 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 5 <sub>1</sub> , 0 <sub>1</sub> - 2 <sub>2</sub> - 0 <sub>2</sub> )	(1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 5 <sub>1</sub> , 0 <sub>2</sub> - 2 <sub>2</sub> - 0 <sub>1</sub> )	{0 <sub>1</sub> , 1 <sub>2</sub> }, {2 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{22}$
(0 <sub>2</sub> ; 4 <sub>1</sub> , 5 <sub>1</sub> , 3 <sub>1</sub> - 2 <sub>2</sub> - 1 <sub>2</sub> ) (1 <sub>2</sub> ; 3 <sub>1</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> )	(0 <sub>2</sub> ; 4 <sub>1</sub> , 5 <sub>1</sub> , 1 <sub>2</sub> - 2 <sub>2</sub> - 3 <sub>1</sub> ) (1 <sub>2</sub> ; 0 <sub>2</sub> , 0 <sub>1</sub> , 1 <sub>1</sub> - 2 <sub>2</sub> - 5 <sub>1</sub> )	{3 <sub>1</sub> , 0 <sub>2</sub> }, {3 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{23}$

Table 4.6: Substitutions in the  $C$ -covering  $R \cup S \cup U$  to obtain different excess graphs for  $n = 14$

In order to obtain the remaining excess graphs, we substitute some trees for others in the covering  $R \cup S$ . These substitutions are illustrated in Table 4.7.

Case 8.  $T = D, n = 13$

The excess graph has two edges by Theorem 1.3.3. Hence, the possible excess graphs are  $K_2^2, P_3$ , and  $2K_2$ . In order to achieve the excess graphs  $P_3$  and  $2K_2$ , label the vertices of  $K_{12}$  with the elements of  $\mathbb{Z}_{12}$ . By Theorem 1.3.2, the leave graph in a maximum  $D$ -packing of  $K_{12}$  has three edges. Moreover, all simple graphs with three edges are achievable as the leave graph, as illustrated in Chapter 3. Let  $R$  be a maximum  $D$ -packing of  $K_{12}$  with the path  $(0, 1, 2, 3)$  as the leave graph. Also let  $S$  be the set consisting of the single tree  $(1; 0, 2, 4 - 3, 5)$  and  $U$  be the set consisting of the single tree  $(1; 4, 0, 2 - 5, 3)$ . Therefore,  $R \cup S$  and  $R \cup U$  are minimum  $D$ -coverings of  $K_{12}$  with the excess graphs  $P_3$  (with the edges  $\{1, 4\}$  and  $\{4, 5\}$ ) and  $2K_2$  (with the edges  $\{0, 5\}$  and  $\{1, 4\}$ ), respectively.

In order to obtain the excess graph  $K_2^2$ , write  $K_{13} = K_{10} \vee K_3$  and label the vertices of  $K_{10} \vee K_3$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.1,  $K_{10}$  has a  $D$ -decomposition,  $R$ . Also the set of vertices  $\{5_1, 6_1, 7_1, 8_1, 9_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2\}$ , and the edges between these two sets, form a graph  $K_{5,3}$ , which has a  $D$ -decomposition,  $S$ , by Lemma 1.3.16. Let  $U$  be the set consisting of the following trees.

$$(1_2; 3_1, 2_1, 0_2 - 2_2, 1_1), (0_2; 2_1, 3_1, 1_2 - 2_2, 4_1), (0_2; 4_1, 1_2, 2_2 - 0_1, 1_1), (2_2; 4_1, 0_1, 1_2 - 0_2, 1_1)$$

Therefore,  $R \cup S \cup U$  forms a minimum  $D$ -covering of  $K_{13}$  with the excess graph  $K_2^2$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used twice.

Case 9.  $T = D, n = 14$

The excess graph has four edges by Theorem 1.3.3. Hence, the possible excess graphs are those illustrated in Figure 4.1. Since a maximum  $D$ -packing of  $K_{14}$  has one edge by Theorem 1.3.2, the excess graphs  $E_2, E_5$ , and  $E_6$  can be obtained as explained in Case 7.

In order to achieve the excess graph  $E_{12}$ , write  $K_{14} = K_{11} \vee K_3$  and label the

New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
$(1_2; 3_1, 4_1, 5_1 - 0_2, 2_2)$ $(1_2; 1_1, 4_1, 5_1 - 2_2, 0_2)$ $(1_2; 7_1, 6_1, 8_1 - 2_2, 0_2)$	$(1_2; 3_1, 5_1, 0_2 - 2_2, 4_1)$ $(1_2; 1_1, 4_1, 0_2 - 2_2, 5_1)$ $(1_2; 7_1, 6_1, 0_2 - 2_2, 8_1)$	$\{4_1, 1_2\}, \{5_1, 1_2\}, \{8_1, 1_2\}, \{0_2, 1_2\}$	$E_1$
$(1_2; 3_1, 4_1, 5_1 - 0_2, 2_2)$ $(1_2; 1_1, 4_1, 5_1 - 2_2, 0_2)$ $(6_1; 2_2, 8_1, 1_2 - 0_2, 7_1)$	$(1_2; 3_1, 5_1, 0_2 - 2_2, 4_1)$ $(1_2; 1_1, 4_1, 0_2 - 2_2, 5_1)$ $(1_2; 7_1, 6_1, 0_2 - 2_2, 8_1)$	$\{4_1, 1_2\}, \{5_1, 1_2\}, \{6_1, 8_1\}, \{0_2, 1_2\}$	$E_3$
$(1_2; 1_1, 4_1, 5_1 - 2_2, 0_2)$ $(5_1; 2_2, 4_1, 1_2 - 0_2, 3_1)$ $(2_2; 1_2, 7_1, 0_2 - 8_1, 4_1)$ $(6_1; 2_2, 0_2, 1_2 - 8_1, 7_1)$	$(1_2; 1_1, 4_1, 0_2 - 2_2, 5_1)$ $(1_2; 3_1, 5_1, 0_2 - 2_2, 4_1)$ $(2_2; 1_2, 7_1, 0_2 - 8_1, 6_1)$ $(1_2; 7_1, 6_1, 0_2 - 2_2, 8_1)$	$\{4_1, 5_1\}, \{4_1, 0_2\}, \{5_1, 1_2\}, \{0_2, 1_2\}$	$E_4$
$(1_2; 1_1, 4_1, 5_1 - 2_2, 0_2)$ $(5_1; 2_2, 0_2, 1_2 - 4_1, 3_1)$ $(1_1; 2_2, 2_1, 0_2 - 1_2, 7_1)$	$(1_2; 1_1, 4_1, 0_2 - 2_2, 5_1)$ $(1_2; 3_1, 5_1, 0_2 - 2_2, 4_1)$ $(0_2; 7_1, 1_1, 1_2 - 2_2, 2_1)$	$\{1_1, 2_1\}, \{5_1, 0_2\}, \{5_1, 1_2\}, \{0_2, 1_2\}$	$E_7$
$(1_2; 1_1, 4_1, 5_1 - 2_2, 0_2)$ $(5_1; 2_2, 0_2, 1_2 - 4_1, 3_1)$ $(0_2; 2_1, 0_1, 3_1 - 1_2, 2_2)$	$(1_2; 1_1, 4_1, 0_2 - 2_2, 5_1)$ $(1_2; 3_1, 5_1, 0_2 - 2_2, 4_1)$ $(0_2; 2_1, 3_1, 1_2 - 2_2, 0_1)$	$\{5_1, 0_2\}, \{5_1, 1_2\}, \{0_1, 0_2\}, \{0_2, 1_2\}$	$E_8$
$(1_2; 3_1, 4_1, 5_1 - 0_2, 2_2)$ $(1_2; 1_1, 4_1, 5_1 - 2_2, 0_2)$ $(0_2; 2_1, 0_1, 3_1 - 1_2, 2_2)$ $(0_2; 7_1, 1_1, 2_1 - 2_2, 1_2)$	$(1_2; 3_1, 5_1, 0_2 - 2_2, 4_1)$ $(1_2; 1_1, 4_1, 0_2 - 2_2, 5_1)$ $(0_2; 2_1, 3_1, 1_2 - 2_2, 0_1)$ $(0_2; 7_1, 1_1, 1_2 - 2_2, 2_1)$	$\{0_1, 0_2\}, \{2_1, 0_2\}, \{4_1, 1_2\}, \{5_1, 1_2\}$	$E_9$
$(1_2; 1_1, 4_1, 5_1 - 2_2, 0_2)$ $(5_1; 2_2, 0_2, 1_2 - 4_1, 3_1)$ $(3_1; 2_2, 0_1, 0_2 - 1_2, 2_1)$ $(1_1; 2_2, 2_1, 0_2 - 1_2, 7_1)$	$(1_2; 1_1, 4_1, 0_2 - 2_2, 5_1)$ $(1_2; 3_1, 5_1, 0_2 - 2_2, 4_1)$ $(0_2; 2_1, 3_1, 1_2 - 2_2, 0_1)$ $(0_2; 7_1, 1_1, 1_2 - 2_2, 2_1)$	$\{1_1, 2_1\}, \{0_1, 3_1\}, \{5_1, 0_2\}, \{5_1, 1_2\}$	$E_{10}$
$(5_1; 2_2, 4_1, 1_2 - 0_2, 3_1)$ $(3_1; 2_2, 0_1, 0_2 - 1_2, 2_1)$ $(1_1; 2_2, 2_1, 0_2 - 1_2, 7_1)$	$(1_2; 3_1, 5_1, 0_2 - 2_2, 4_1)$ $(0_2; 2_1, 3_1, 1_2 - 2_2, 0_1)$ $(0_2; 7_1, 1_1, 1_2 - 2_2, 2_1)$	$\{1_1, 2_1\}, \{0_1, 3_1\}, \{4_1, 5_1\}, \{0_2, 1_2\}$	$E_{11}$
$(1_1; 2_2, 0_2, 1_2 - 7_1, 2_1)$	$(0_2; 7_1, 1_1, 1_2 - 2_2, 2_1)$	$\{1_1, 1_2\}, \{0_2, 1_2\}$ (3 times)	$E_{13}$
$(1_1; 2_2, 0_2, 2_1 - 7_1, 1_2)$	$(0_2; 7_1, 1_2, 1_1 - 2_1, 2_2)$	$\{1_1, 2_1\}, \{0_2, 1_2\}$ (3 times)	$E_{14}$
$(0_2; 7_1, 1_1, 2_1 - 2_2, 1_2)$ $(2_2; 0_1, 2_1, 8_1 - 0_2, 1_2)$ $(0_2; 2_1, 0_1, 3_1 - 1_2, 2_2)$	$(0_2; 7_1, 1_1, 1_2 - 2_2, 2_1)$ $(2_2; 2_1, 0_1, 8_1 - 0_2, 1_2)$ $(0_2; 2_1, 1_2, 3_1 - 0_1, 2_2)$	$\{2_1, 0_2\}$ (twice), $\{0_2, 1_2\}$ (twice)	$E_{15}$
$(5_1; 2_2, 4_1, 1_2 - 0_2, 3_1)$ $(4_1; 2_2, 5_1, 1_2 - 0_2, 1_1)$	$(1_2; 3_1, 5_1, 0_2 - 2_2, 4_1)$ $(1_2; 1_1, 4_1, 0_2 - 2_2, 5_1)$	$\{4_1, 5_1\}$ (twice), $\{0_2, 1_2\}$ (twice)	$E_{16}$
$(5_1; 2_2, 4_1, 1_2 - 0_2, 3_1)$ $(1_2; 1_1, 4_1, 5_1 - 2_2, 0_2)$	$(1_2; 3_1, 5_1, 0_2 - 2_2, 4_1)$ $(1_2; 1_1, 4_1, 0_2 - 2_2, 5_1)$	$\{4_1, 5_1\}, \{5_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$E_{17}$
$(1_2; 3_1, 4_1, 5_1 - 0_2, 2_2)$ $(3_1; 2_2, 0_1, 0_2 - 1_2, 2_1)$	$(1_2; 3_1, 5_1, 0_2 - 2_2, 4_1)$ $(0_2; 2_1, 3_1, 1_2 - 2_2, 0_1)$	$\{0_1, 3_1\}, \{4_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$E_{18}$
$(4_1; 2_2, 5_1, 1_2 - 0_2, 1_1)$ $(6_1; 2_2, 8_1, 1_2 - 0_2, 7_1)$	$(1_2; 1_1, 4_1, 0_2 - 2_2, 5_1)$ $(1_2; 7_1, 6_1, 0_2 - 2_2, 8_1)$	$\{4_1, 5_1\}, \{6_1, 8_1\}, \{0_2, 1_2\}$ (twice)	$E_{19}$
$(1_2; 7_1, 6_1, 8_1 - 2_2, 0_2)$ $(2_2; 2_1, 0_1, 8_1 - 0_2, 5_1)$ $(4_1; 2_2, 5_1, 1_2 - 0_2, 1_1)$	$(1_2; 7_1, 6_1, 0_2 - 2_2, 8_1)$ $(2_2; 2_1, 0_1, 8_1 - 0_2, 1_2)$ $(1_2; 1_1, 4_1, 0_2 - 2_2, 5_1)$	$\{4_1, 5_1\}, \{5_1, 8_1\}, \{0_2, 1_2\}$ (twice)	$E_{20}$
$(0_2; 2_1, 0_1, 3_1 - 1_2, 2_2)$ $(1_2; 3_1, 4_1, 5_1 - 0_2, 2_2)$	$(0_2; 2_1, 3_1, 1_2 - 2_2, 0_1)$ $(1_2; 3_1, 5_1, 0_2 - 2_2, 4_1)$	$\{0_1, 0_2\}, \{4_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$E_{21}$
$(1_2; 3_1, 4_1, 5_1 - 0_2, 2_2)$ $(1_2; 1_1, 4_1, 5_1 - 2_2, 0_2)$	$(1_2; 3_1, 5_1, 0_2 - 2_2, 4_1)$ $(1_2; 1_1, 4_1, 0_2 - 2_2, 5_1)$	$\{4_1, 1_2\}, \{5_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$E_{22}$
$(1_2; 1_1, 4_1, 5_1 - 2_2, 0_2)$ $(5_1; 2_2, 0_2, 1_2 - 4_1, 3_1)$	$(1_2; 1_1, 4_1, 0_2 - 2_2, 5_1)$ $(1_2; 3_1, 5_1, 0_2 - 2_2, 4_1)$	$\{5_1, 0_2\}, \{5_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$E_{23}$

Table 4.7: Substitutions in the  $D$ -covering  $R \cup S$  to obtain different excess graphs for  $n = 12$

vertices of  $K_{11} \vee K_3$  with a  $(\mathbb{Z}_{11}, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.1,  $K_{11}$  has a  $D$ -decomposition,  $R$ . Let  $S$  be the set consisting of the following trees.

$$\begin{aligned} &(0_2; 0_1, 1_1, 1_2 - 2_2, 2_1), (0_2; 2_1, 3_1, 1_2 - 2_2, 4_1), (1_2; 1_1, 0_1, 0_2 - 2_2, 4_1), \\ &(1_2; 3_1, 5_1, 0_2 - 2_2, 9_1), (1_2; 10_1, 7_1, 0_2 - 2_2, 8_1), (2_2; 2_1, 0_2, 1_2 - 5_1, 8_1), \\ &(2_2; 4_1, 9_1, 10_1 - 1_2, 0_2), (6_1; 1_2, 0_2, 2_2 - 7_1, 8_1) \end{aligned}$$

Therefore,  $R \cup S$  forms a minimum  $D$ -covering of  $K_{14}$  with the excess graph  $E_{12}$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used four times.

As in the previous cases, the rest of the excess graphs will be achieved by substituting some trees for others in the covering  $R \cup S$ . These substitutions are illustrated in Table 4.8.

Case 10.  $T = E, n = 12$

The excess graph has four edges by Theorem 1.3.3. Hence, the possible excess graphs are those shown in Figure 4.1. The leave graph in a maximum  $E$ -packing of  $K_{12}$  has one edge by Theorem 1.3.2. In order to obtain the excess graphs  $E_2$  and  $E_9$ , label the vertices of  $K_{12}$  with the elements of  $\mathbb{Z}_{12}$  and let  $R$  be a maximum  $E$ -packing of  $K_{12}$  with the edge  $\{0, 1\}$  as the leave graph. Also let  $S$  be the set consisting of the single tree  $(1 - 0, 2; 3 - 4, 5)$  and  $U$  be the set consisting of the single tree  $(0 - 2, 3; 1 - 4, 5)$ . Therefore,  $R \cup S$  and  $R \cup U$  form minimum  $E$ -coverings of  $K_{12}$  with the excess graphs  $E_2$  (with the edges  $\{1, 2\}, \{1, 3\}, \{3, 4\}$ , and  $\{3, 5\}$ ) and  $E_9$  (with the edges  $\{0, 2\}, \{0, 3\}, \{1, 4\}$ , and  $\{1, 5\}$ ), respectively.

In order to achieve the excess graph  $E_{12}$ , write  $K_{12} = K_9 \vee K_3$  and label the vertices of  $K_9 \vee K_3$  with a  $(\mathbb{Z}_9, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.2, the leave graph in a maximum  $E$ -packing of  $K_9$  has one edge. Let  $R$  be a maximum  $E$ -packing of  $K_9$  with the edge  $\{7_1, 8_1\}$  as the leave graph. Also let  $S$  be the set consisting of the following trees.

$$\begin{aligned} &(0_2 - 0_1, 1_1; 1_2 - 2_1, 3_1), (1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1), (0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1), \\ &(1_2 - 1_1, 0_2; 8_1 - 7_1, 2_2), (0_2 - 2_1, 3_1; 1_2 - 4_1, 5_1), (7_1 - 0_2, 1_2; 2_2 - 4_1, 5_1), \\ &(6_1 - 1_2, 2_2; 0_2 - 5_1, 8_1) \end{aligned}$$

Therefore,  $R \cup S$  forms a minimum  $E$ -covering of  $K_{12}$  with the excess graph  $E_{12}$ . The

New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
$(1_2; 1_1, 0_1, 4_1 - 2_2, 0_2)$ $(1_2; 3_1, 5_1, 9_1 - 2_2, 0_2)$ $(1_2; 10_1, 7_1, 8_1 - 2_2, 0_2)$	$(1_2; 1_1, 0_1, 0_2 - 2_2, 4_1)$ $(1_2; 3_1, 5_1, 0_2 - 2_2, 9_1)$ $(1_2; 10_1, 7_1, 0_2 - 2_2, 8_1)$	$\{4_1, 1_2\}, \{8_1, 1_2\}, \{9_1, 1_2\}, \{0_2, 1_2\}$	$E_1$
$(1_2; 1_1, 0_1, 4_1 - 2_2, 0_2)$ $(1_2; 3_1, 5_1, 9_1 - 2_2, 0_2)$ $(7_1; 2_2, 8_1, 1_2 - 0_2, 10_1)$	$(1_2; 1_1, 0_1, 0_2 - 2_2, 4_1)$ $(1_2; 3_1, 5_1, 0_2 - 2_2, 9_1)$ $(1_2; 10_1, 7_1, 0_2 - 2_2, 8_1)$	$\{7_1, 8_1\}, \{4_1, 1_2\}, \{9_1, 1_2\}, \{0_2, 1_2\}$	$E_3$
$(3_1; 2_2, 4_1, 0_2 - 1_2, 2_1)$ $(1_2; 1_1, 0_1, 4_1 - 2_2, 0_2)$ $(5_1; 2_2, 0_2, 1_2 - 9_1, 3_1)$ $(2_2; 2_1, 0_2, 1_2 - 3_1, 8_1)$	$(0_2; 2_1, 3_1, 1_2 - 2_2, 4_1)$ $(1_2; 1_1, 0_1, 0_2 - 2_2, 4_1)$ $(1_2; 3_1, 5_1, 0_2 - 2_2, 9_1)$ $(2_2; 2_1, 0_2, 1_2 - 5_1, 8_1)$	$\{3_1, 4_1\}, \{3_1, 0_2\}, \{4_1, 1_2\}, \{0_2, 1_2\}$	$E_4$
$(1_2; 1_1, 0_1, 4_1 - 2_2, 0_2)$ $(0_2; 2_1, 3_1, 4_1 - 2_2, 1_2)$ $(1_1; 2_2, 2_1, 0_2 - 1_2, 0_1)$	$(1_2; 1_1, 0_1, 0_2 - 2_2, 4_1)$ $(0_2; 2_1, 3_1, 1_2 - 2_2, 4_1)$ $(0_2; 0_1, 1_1, 1_2 - 2_2, 2_1)$	$\{1_1, 2_1\}, \{4_1, 0_2\}, \{4_1, 1_2\}, \{0_2, 1_2\}$	$E_7$
$(1_2; 1_1, 0_1, 4_1 - 2_2, 0_2)$ $(0_2; 2_1, 3_1, 4_1 - 2_2, 1_2)$ $(0_2; 0_1, 1_1, 2_1 - 2_2, 1_2)$	$(1_2; 1_1, 0_1, 0_2 - 2_2, 4_1)$ $(0_2; 2_1, 3_1, 1_2 - 2_2, 4_1)$ $(0_2; 0_1, 1_1, 1_2 - 2_2, 2_1)$	$\{2_1, 0_2\}, \{4_1, 0_2\}, \{4_1, 1_2\}, \{0_2, 1_2\}$	$E_8$
$(3_1; 2_2, 4_1, 0_2 - 1_2, 2_1)$ $(0_1; 2_2, 4_1, 1_2 - 0_2, 1_1)$ $(1_2; 3_1, 5_1, 9_1 - 2_2, 0_2)$	$(0_2; 2_1, 3_1, 1_2 - 2_2, 4_1)$ $(1_2; 1_1, 0_1, 0_2 - 2_2, 4_1)$ $(1_2; 3_1, 5_1, 0_2 - 2_2, 9_1)$	$\{0_1, 4_1\}, \{3_1, 4_1\}, \{9_1, 1_2\}, \{0_2, 1_2\}$	$E_9$
$(3_1; 2_2, 4_1, 0_2 - 1_2, 2_1)$ $(1_2; 3_1, 5_1, 9_1 - 2_2, 0_2)$ $(1_1; 2_2, 2_1, 0_2 - 1_2, 0_1)$	$(0_2; 2_1, 3_1, 1_2 - 2_2, 4_1)$ $(1_2; 3_1, 5_1, 0_2 - 2_2, 9_1)$ $(0_2; 0_1, 1_1, 1_2 - 2_2, 2_1)$	$\{1_1, 2_1\}, \{3_1, 4_1\}, \{9_1, 1_2\}, \{0_2, 1_2\}$	$E_{10}$
$(1_1; 2_2, 2_1, 0_2 - 1_2, 0_1)$ $(3_1; 2_2, 4_1, 0_2 - 1_2, 2_1)$ $(5_1; 2_2, 9_1, 1_2 - 0_2, 3_1)$	$(0_2; 0_1, 1_1, 1_2 - 2_2, 2_1)$ $(0_2; 2_1, 3_1, 1_2 - 2_2, 4_1)$ $(1_2; 3_1, 5_1, 0_2 - 2_2, 9_1)$	$\{1_1, 2_1\}, \{3_1, 4_1\}, \{5_1, 9_1\}, \{0_2, 1_2\}$	$E_{11}$
$(0_2; 0_1, 1_1, 2_1 - 2_2, 1_2)$	$(0_2; 0_1, 1_1, 1_2 - 2_2, 2_1)$	$\{2_1, 0_2\}, \{0_2, 1_2\}$ (3 times)	$E_{13}$
$(1_1; 2_2, 2_1, 0_2 - 1_2, 0_1)$	$(0_2; 0_1, 1_1, 1_2 - 2_2, 2_1)$	$\{1_1, 2_1\}, \{0_2, 1_2\}$ (3 times)	$E_{14}$
$(1_2; 3_1, 5_1, 9_1 - 2_2, 0_2)$ $(1_2; 10_1, 7_1, 8_1 - 2_2, 0_2)$ $(2_2; 2_1, 0_2, 1_2 - 5_1, 9_1)$	$(1_2; 3_1, 5_1, 0_2 - 2_2, 9_1)$ $(1_2; 10_1, 7_1, 0_2 - 2_2, 8_1)$ $(2_2; 2_1, 0_2, 1_2 - 5_1, 8_1)$	$\{9_1, 1_2\}$ (twice), $\{0_2, 1_2\}$ (twice)	$E_{15}$
$(1_2; 3_1, 5_1, 9_1 - 2_2, 0_2)$ $(1_2; 10_1, 7_1, 8_1 - 2_2, 0_2)$ $(2_2; 2_1, 0_2, 1_2 - 5_1, 9_1)$ $(0_2; 0_1, 1_1, 2_1 - 2_2, 1_2)$ $(0_2; 2_1, 3_1, 4_1 - 2_2, 1_2)$ $(1_2; 1_1, 0_1, 0_2 - 2_2, 2_1)$	$(1_2; 3_1, 5_1, 0_2 - 2_2, 9_1)$ $(1_2; 10_1, 7_1, 0_2 - 2_2, 8_1)$ $(2_2; 2_1, 0_2, 1_2 - 5_1, 8_1)$ $(0_2; 0_1, 1_1, 1_2 - 2_2, 2_1)$ $(0_2; 2_1, 3_1, 1_2 - 2_2, 4_1)$ $(1_2; 1_1, 0_1, 0_2 - 2_2, 4_1)$	$\{2_1, 0_2\}$ (twice), $\{9_1, 1_2\}$ (twice)	$E_{16}$
$(1_2; 1_1, 0_1, 4_1 - 2_2, 0_2)$ $(3_1; 2_2, 4_1, 0_2 - 1_2, 2_1)$	$(1_2; 1_1, 0_1, 0_2 - 2_2, 4_1)$ $(0_2; 2_1, 3_1, 1_2 - 2_2, 4_1)$	$\{3_1, 4_1\}, \{4_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$E_{17}$
$(1_2; 1_1, 0_1, 4_1 - 2_2, 0_2)$ $(1_1; 2_2, 2_1, 0_2 - 1_2, 0_1)$	$(1_2; 1_1, 0_1, 0_2 - 2_2, 4_1)$ $(0_2; 0_1, 1_1, 1_2 - 2_2, 2_1)$	$\{1_1, 2_1\}, \{4_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$E_{18}$
$(3_1; 2_2, 4_1, 0_2 - 1_2, 2_1)$ $(1_1; 2_2, 2_1, 0_2 - 1_2, 0_1)$	$(0_2; 2_1, 3_1, 1_2 - 2_2, 4_1)$ $(0_2; 0_1, 1_1, 1_2 - 2_2, 2_1)$	$\{1_1, 2_1\}, \{3_1, 4_1\}, \{0_2, 1_2\}$ (twice)	$E_{19}$
$(3_1; 2_2, 4_1, 0_2 - 1_2, 2_1)$ $(0_1; 2_2, 4_1, 1_2 - 0_2, 1_1)$	$(0_2; 2_1, 3_1, 1_2 - 2_2, 4_1)$ $(1_2; 1_1, 0_1, 0_2 - 2_2, 4_1)$	$\{0_1, 4_1\}, \{3_1, 4_1\}, \{0_2, 1_2\}$ (twice)	$E_{20}$
$(1_2; 1_1, 0_1, 4_1 - 2_2, 0_2)$ $(0_2; 0_1, 1_1, 2_1 - 2_2, 1_2)$	$(1_2; 1_1, 0_1, 0_2 - 2_2, 4_1)$ $(0_2; 0_1, 1_1, 1_2 - 2_2, 2_1)$	$\{2_1, 0_2\}, \{4_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$E_{21}$
$(1_2; 1_1, 0_1, 4_1 - 2_2, 0_2)$ $(1_2; 10_1, 7_1, 8_1 - 2_2, 0_2)$	$(1_2; 1_1, 0_1, 0_2 - 2_2, 4_1)$ $(1_2; 10_1, 7_1, 0_2 - 2_2, 8_1)$	$\{4_1, 1_2\}, \{8_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$E_{22}$
$(1_2; 1_1, 0_1, 4_1 - 2_2, 0_2)$ $(0_2; 2_1, 3_1, 4_1 - 2_2, 1_2)$	$(1_2; 1_1, 0_1, 0_2 - 2_2, 4_1)$ $(0_2; 2_1, 3_1, 1_2 - 2_2, 4_1)$	$\{4_1, 0_2\}, \{4_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$E_{23}$

Table 4.8: Substitutions in the  $D$ -covering  $R \cup S$  to obtain different excess graphs for  $n = 14$

edges of the excess graph are the edges  $\{0_2, 1_2\}$  used four times.

In order to obtain the rest of the excess graphs, we substitute some trees for others in the covering  $R \cup S$ . These substitutions are indicated in Table 4.9.

Case 11.  $T = E, n = 13$

The excess graph has two edges by Theorem 1.3.3. Hence, the possible excess graphs are  $K_2^2, P_3$ , and  $2K_2$ . The leave graph in a maximum  $E$ -packing of  $K_{13}$  has three edges by Theorem 1.3.2. All simple graphs with three edges can be achieved as the leave graph in  $E$ -packings of  $K_{13}$  as illustrated in Chapter 3. Label the vertices of  $K_{13}$  with the elements of  $\mathbb{Z}_{13}$ . Let  $R$  and  $S$  be maximum  $E$ -packings of  $K_{13}$  with the leave graphs  $(0; 1, 2, 3)$  and  $(0, 1, 2, 3)$ , respectively. Also let  $U$  be the set consisting of the single tree  $(0 - 1, 2; 3 - 4, 5)$  and  $V$  be the set consisting of the single tree  $(1 - 0, 4; 2 - 3, 5)$ . Therefore,  $R \cup U$  and  $S \cup V$  are minimum  $E$ -coverings of  $K_{13}$  with the excess graphs  $P_3$  (with the edges  $\{3, 4\}$  and  $\{3, 5\}$ ) and  $2K_2$  (with the edges  $\{1, 4\}$  and  $\{2, 5\}$ ), respectively.

In order to achieve the excess graph  $K_2^2$ , write  $K_{13} = K_{10} \vee K_3$  and label the vertices of  $K_{10} \vee K_3$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_3)$ -labeling. By Theorem 1.3.1,  $K_{10}$  has an  $E$ -decomposition,  $R$ . Let  $S$  be the set consisting of the following trees.

$$\begin{aligned} &(0_2 - 0_1, 1_1; 1_2 - 2_1, 3_1), (0_2 - 2_1, 3_1; 1_2 - 0_1, 1_1), (0_2 - 4_1, 5_1; 1_2 - 6_1, 7_1), \\ &(1_2 - 4_1, 5_1; 2_2 - 0_1, 1_1), (0_2 - 6_1, 7_1; 2_2 - 2_1, 3_1), (8_1 - 0_2, 1_2; 2_2 - 4_1, 5_1), \\ &(9_1 - 0_2, 1_2; 2_2 - 6_1, 7_1) \end{aligned}$$

Therefore,  $R \cup S$  forms a minimum  $E$ -covering of  $K_{13}$  with the excess graph  $K_2^2$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used twice.

Case 12.  $T = E, n = 14$

The excess graph has four edges by Theorem 1.3.3. Hence, the possible excess graphs are those shown in Figure 4.1. Since the leave graph in a maximum  $E$ -packing of  $K_{14}$  has one edge by Theorem 1.3.2, the excess graphs  $E_2$  and  $E_9$  can be obtained in a similar way as in Case 10.

In order to achieve the excess graph  $E_{12}$ , write  $K_{14} = K_{10} \vee K_4$  and label the



New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
$(1_2 - 2_1, 3_1; 2_2 - 4_1, 5_1)$ $(7_1 - 1_2, 2_2; 0_2 - 0_1, 1_1)$ $(1_2 - 0_1, 3_1; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 4_1)$	$(1_2 - 2_1, 3_1; 0_2 - 0_1, 1_1)$ $(7_1 - 0_2, 1_2; 2_2 - 4_1, 5_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 0_2)$	$\{3_1, 1_2\}, \{4_1, 1_2\}, \{0_2, 1_2\}, \{1_2, 2_2\}$	$E_1$
$(1_2 - 0_1, 3_1; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 4_1)$ $(0_2 - 4_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 8_1, 1_2; 2_2 - 4_1, 5_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 0_2)$ $(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 2_2 - 4_1, 5_1)$	$\{7_1, 8_1\}, \{3_1, 1_2\}, \{4_1, 1_2\}, \{0_2, 1_2\}$	$E_3$
$(0_2 - 4_1, 6_1; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 7_1; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 6_1)$ $(6_1 - 7_1, 2_2; 0_2 - 5_1, 8_1)$	$(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 0_2)$ $(6_1 - 1_2, 2_2; 0_2 - 5_1, 8_1)$	$\{6_1, 7_1\}, \{6_1, 0_2\}, \{7_1, 1_2\}, \{0_2, 1_2\}$	$E_4$
$(0_2 - 1_1, 4_1; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 3_1; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 6_1)$ $(6_1 - 7_1, 2_2; 0_2 - 5_1, 8_1)$	$(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 0_2)$ $(6_1 - 1_2, 2_2; 0_2 - 5_1, 8_1)$	$\{6_1, 7_1\}, \{1_1, 0_2\}, \{3_1, 1_2\}, \{0_2, 1_2\}$	$E_5$
$(0_2 - 1_1, 4_1; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 6_1; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 6_1)$ $(6_1 - 7_1, 2_2; 0_2 - 5_1, 8_1)$	$(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 0_2)$ $(6_1 - 1_2, 2_2; 0_2 - 5_1, 8_1)$	$\{6_1, 7_1\}, \{1_1, 0_2\}, \{6_1, 1_2\}, \{0_2, 1_2\}$	$E_6$
$(0_2 - 4_1, 5_1; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 5_1; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 6_1)$ $(6_1 - 7_1, 2_2; 0_2 - 5_1, 8_1)$	$(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 0_2)$ $(6_1 - 1_2, 2_2; 0_2 - 5_1, 8_1)$	$\{6_1, 7_1\}, \{5_1, 0_2\}, \{5_1, 1_2\}, \{0_2, 1_2\}$	$E_7$
$(0_2 - 4_1, 5_1; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 5_1; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 6_1)$	$(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 0_2)$	$\{5_1, 0_2\}, \{5_1, 1_2\}, \{6_1, 1_2\}, \{0_2, 1_2\}$	$E_8$
$(0_2 - 1_1, 4_1; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 3_1; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 6_1)$ $(6_1 - 7_1, 2_2; 0_2 - 5_1, 8_1)$ $(1_2 - 2_1, 3_1; 2_2 - 4_1, 5_1)$ $(7_1 - 1_2, 2_2; 0_2 - 0_1, 1_1)$	$(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 0_2)$ $(6_1 - 1_2, 2_2; 0_2 - 5_1, 8_1)$ $(0_2 - 0_1, 1_1; 1_2 - 2_1, 3_1)$ $(7_1 - 0_2, 1_2; 2_2 - 4_1, 5_1)$	$\{6_1, 7_1\}, \{1_1, 0_2\}, \{3_1, 1_2\}, \{1_2, 2_2\}$	$E_{10}$
$(0_2 - 4_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 8_1, 2_2; 1_2 - 2_1, 3_1)$ $(0_2 - 0_1, 1_1; 2_2 - 4_1, 5_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 6_1)$ $(6_1 - 4_1, 2_2; 0_2 - 5_1, 8_1)$ $(1_2 - 0_1, 3_1; 2_2 - 1_1, 2_1)$	$(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 2_2 - 4_1, 5_1)$ $(0_2 - 0_1, 1_1; 1_2 - 2_1, 3_1)$ $(8_1 - 7_1, 2_2; 1_2 - 1_1, 0_2)$ $(6_1 - 1_2, 2_2; 0_2 - 5_1, 8_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$	$\{4_1, 6_1\}, \{7_1, 8_1\}, \{3_1, 1_2\}, \{0_2, 2_2\}$	$E_{11}$
$(0_2 - 1_1, 4_1; 2_2 - 0_1, 3_1)$	$(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$	$\{1_1, 0_2\}, \{0_2, 1_2\}$ (3 times)	$E_{13}$
$(1_2 - 0_1, 7_1; 2_2 - 1_1, 2_1)$ $(7_1 - 8_1, 0_2; 2_2 - 4_1, 5_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(7_1 - 0_2, 1_2; 2_2 - 4_1, 5_1)$	$\{7_1, 8_1\}, \{0_2, 1_2\}$ (3 times)	$E_{14}$
$(1_2 - 0_1, 3_1; 2_2 - 1_1, 2_1)$ $(1_2 - 1_1, 3_1; 8_1 - 7_1, 2_2)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(1_2 - 1_1, 0_2; 8_1 - 7_1, 2_2)$	$\{3_1, 1_2\}$ (twice), $\{0_2, 1_2\}$ (twice)	$E_{15}$
$(1_2 - 0_1, 6_1; 2_2 - 1_1, 2_1)$ $(6_1 - 7_1, 2_2; 0_2 - 5_1, 8_1)$ $(0_2 - 4_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 6_1, 1_2; 2_2 - 4_1, 5_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(6_1 - 1_2, 2_2; 0_2 - 5_1, 8_1)$ $(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 2_2 - 4_1, 5_1)$	$\{6_1, 7_1\}$ (twice), $\{0_2, 1_2\}$ (twice)	$E_{16}$
$(0_2 - 1_1, 4_1; 2_2 - 0_1, 3_1)$ $(7_1 - 1_1, 0_2; 2_2 - 4_1, 5_1)$ $(1_2 - 0_1, 7_1; 2_2 - 1_1, 2_1)$	$(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 2_2 - 4_1, 5_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$	$\{1_1, 7_1\}, \{1_1, 0_2\}, \{0_2, 1_2\}$ (twice)	$E_{17}$

New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
$(0_2 - 1_1, 4_1; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 7_1; 2_2 - 1_1, 2_1)$ $(7_1 - 8_1, 0_2; 2_2 - 4_1, 5_1)$	$(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(7_1 - 0_2, 1_2; 2_2 - 4_1, 5_1)$	$\{7_1, 8_1\}, \{1_1, 0_2\}, \{0_2, 1_2\}$ (twice)	$E_{18}$
$(1_2 - 0_1, 6_1; 2_2 - 1_1, 2_1)$ $(6_1 - 4_1, 2_2; 0_2 - 5_1, 8_1)$ $(0_2 - 4_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 8_1, 1_2; 2_2 - 4_1, 5_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(6_1 - 1_2, 2_2; 0_2 - 5_1, 8_1)$ $(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 2_2 - 4_1, 5_1)$	$\{4_1, 6_1\}, \{7_1, 8_1\}, \{0_2, 1_2\}$ (twice)	$E_{19}$
$(1_2 - 0_1, 6_1; 2_2 - 1_1, 2_1)$ $(6_1 - 7_1, 2_2; 0_2 - 5_1, 8_1)$ $(0_2 - 4_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 8_1, 1_2; 2_2 - 4_1, 5_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(6_1 - 1_2, 2_2; 0_2 - 5_1, 8_1)$ $(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 2_2 - 4_1, 5_1)$	$\{6_1, 7_1\}, \{7_1, 8_1\}, \{0_2, 1_2\}$ (twice)	$E_{20}$
$(0_2 - 4_1, 5_1; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 6_1; 2_2 - 1_1, 2_1)$	$(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$	$\{5_1, 0_2\}, \{6_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$E_{21}$
$(1_2 - 0_1, 3_1; 2_2 - 1_1, 2_1)$ $(1_2 - 1_1, 4_1; 8_1 - 7_1, 2_2)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(1_2 - 1_1, 0_2; 8_1 - 7_1, 2_2)$	$\{3_1, 1_2\}, \{4_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$E_{22}$
$(0_2 - 4_1, 5_1; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 5_1; 2_2 - 1_1, 2_1)$	$(0_2 - 4_1, 1_2; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$	$\{5_1, 0_2\}, \{5_1, 1_2\}, \{0_2, 1_2\}$ (twice)	$E_{23}$

Table 4.9: Substitutions in the  $E$ -covering  $R \cup S$  to obtain different excess graphs for  $n = 12$

vertices of  $K_{10} \vee K_4$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_4)$ -labeling. By Theorem 1.3.1,  $K_{10}$  has an  $E$ -decomposition,  $R$ . Let  $S$  be the set consisting of the following trees.

$$\begin{aligned}
&(0_2 - 0_1, 1_1; 1_2 - 2_1, 3_1), (1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1), (0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1), \\
&(0_2 - 3_1, 1_2; 3_2 - 0_1, 1_1), (0_2 - 4_1, 5_1; 1_2 - 1_1, 3_2), (2_2 - 4_1, 5_1; 3_2 - 2_1, 3_1), \\
&(9_1 - 0_2, 1_2; 3_2 - 4_1, 5_1), (8_1 - 0_2, 1_2; 2_2 - 7_1, 9_1), (7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1), \\
&(6_1 - 0_2, 2_2; 1_2 - 4_1, 5_1)
\end{aligned}$$

Therefore,  $R \cup S$  forms a minimum  $E$ -covering of  $K_{14}$  with the excess graph  $E_{12}$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used four times.

In order to obtain the remaining excess graphs, we substitute some trees for others in the covering  $R \cup S$ . Table 4.10 illustrates these substitutions.

Case 13.  $T = P_6, n = 12$

The excess graph has four edges by Theorem 1.3.3. Hence the possible excess graphs are those illustrated in Figure 4.1. A maximum  $P_6$ -packing of  $K_{12}$  has one edge by Theorem 1.3.2. Label the vertices of  $K_{12}$  with the elements of  $\mathbb{Z}_{12}$ . Also let  $R$  be a maximum  $P_6$ -packing of  $K_{12}$  with the edge  $\{0, 1\}$  as the leave graph,

New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
$(1_2 - 0_1, 7_1; 2_2 - 1_1, 2_1)$ $(7_1 - 0_2, 2_2; 3_2 - 6_1, 8_1)$ $(8_1 - 0_2, 1_2; 2_2 - 4_1, 9_1)$ $(2_2 - 5_1, 0_2; 3_2 - 2_1, 3_1)$ $(0_2 - 1_1, 2_1; 2_2 - 0_1, 3_1)$ $(0_2 - 2_1, 3_1; 3_2 - 0_1, 1_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$ $(8_1 - 0_2, 1_2; 2_2 - 7_1, 9_1)$ $(2_2 - 4_1, 5_1; 3_2 - 2_1, 3_1)$ $(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(0_2 - 3_1, 1_2; 3_2 - 0_1, 1_1)$	$\{1_1, 0_2\}, \{2_1, 0_2\}, \{0_2, 1_2\}, \{0_2, 2_2\}$	$E_1$
$(0_2 - 1_1, 2_1; 2_2 - 0_1, 3_1)$ $(0_2 - 2_1, 3_1; 3_2 - 0_1, 1_1)$ $(1_2 - 0_1, 7_1; 2_2 - 1_1, 2_1)$ $(7_1 - 5_1, 0_2; 3_2 - 6_1, 8_1)$	$(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(0_2 - 3_1, 1_2; 3_2 - 0_1, 1_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$	$\{5_1, 7_1\}, \{1_1, 0_2\}, \{2_1, 0_2\}, \{0_2, 1_2\}$	$E_3$
$(0_2 - 2_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 5_1, 1_2; 3_2 - 6_1, 8_1)$ $(0_2 - 3_1, 5_1; 3_2 - 0_1, 1_1)$ $(1_2 - 0_1, 7_1; 2_2 - 1_1, 2_1)$	$(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$ $(0_2 - 3_1, 1_2; 3_2 - 0_1, 1_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$	$\{5_1, 7_1\}, \{5_1, 0_2\}, \{7_1, 1_2\}, \{0_2, 1_2\}$	$E_4$
$(0_2 - 2_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 5_1, 1_2; 3_2 - 6_1, 8_1)$ $(1_2 - 0_1, 3_1; 2_2 - 1_1, 2_1)$ $(0_2 - 3_1, 4_1; 3_2 - 0_1, 1_1)$	$(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(0_2 - 3_1, 1_2; 3_2 - 0_1, 1_1)$	$\{5_1, 7_1\}, \{3_1, 1_2\}, \{4_1, 0_2\}, \{0_2, 1_2\}$	$E_5$
$(0_2 - 2_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 5_1, 1_2; 3_2 - 6_1, 8_1)$ $(1_2 - 0_1, 3_1; 2_2 - 1_1, 2_1)$ $(0_2 - 3_1, 5_1; 3_2 - 0_1, 1_1)$	$(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(0_2 - 3_1, 1_2; 3_2 - 0_1, 1_1)$	$\{5_1, 7_1\}, \{3_1, 1_2\}, \{5_1, 0_2\}, \{0_2, 1_2\}$	$E_6$
$(0_2 - 2_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 5_1, 1_2; 3_2 - 6_1, 8_1)$ $(1_2 - 0_1, 4_1; 2_2 - 1_1, 2_1)$ $(0_2 - 3_1, 4_1; 3_2 - 0_1, 1_1)$	$(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(0_2 - 3_1, 1_2; 3_2 - 0_1, 1_1)$	$\{5_1, 7_1\}, \{4_1, 1_2\}, \{4_1, 0_2\}, \{0_2, 1_2\}$	$E_7$
$(0_2 - 2_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 5_1, 1_2; 3_2 - 6_1, 8_1)$ $(1_2 - 0_1, 5_1; 2_2 - 1_1, 2_1)$ $(0_2 - 3_1, 5_1; 3_2 - 0_1, 1_1)$	$(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(0_2 - 3_1, 1_2; 3_2 - 0_1, 1_1)$	$\{5_1, 7_1\}, \{5_1, 0_2\}, \{5_1, 1_2\}, \{0_2, 1_2\}$	$E_8$
$(1_2 - 0_1, 9_1; 2_2 - 1_1, 2_1)$ $(9_1 - 8_1, 0_2; 3_2 - 4_1, 5_1)$ $(0_2 - 2_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 5_1, 1_2; 3_2 - 6_1, 8_1)$ $(0_2 - 2_1, 3_1; 3_2 - 0_1, 1_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(9_1 - 0_2, 1_2; 3_2 - 4_1, 5_1)$ $(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$ $(0_2 - 2_1, 1_2; 3_2 - 0_1, 1_1)$	$\{5_1, 7_1\}, \{8_1, 9_1\}, \{3_1, 0_2\}, \{0_2, 1_2\}$	$E_{10}$
$(1_2 - 0_1, 9_1; 2_2 - 1_1, 2_1)$ $(9_1 - 4_1, 0_2; 3_2 - 4_1, 5_1)$ $(0_2 - 2_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 5_1, 1_2; 3_2 - 6_1, 8_1)$ $(0_2 - 3_1, 8_1; 3_2 - 0_1, 1_1)$ $(8_1 - 6_1, 1_2; 2_2 - 7_1, 9_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(9_1 - 0_2, 1_2; 3_2 - 4_1, 5_1)$ $(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$ $(0_2 - 3_1, 1_2; 3_2 - 0_1, 1_1)$ $(8_1 - 0_2, 1_2; 2_2 - 7_1, 9_1)$	$\{4_1, 9_1\}, \{5_1, 7_1\}, \{6_1, 8_1\}, \{0_2, 1_2\}$	$E_{11}$
$(1_2 - 0_1, 3_1; 2_2 - 1_1, 2_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$	$\{3_1, 1_2\}, \{0_2, 1_2\}$ (3 times)	$E_{13}$
$(0_2 - 2_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 5_1, 1_2; 3_2 - 6_1, 8_1)$	$(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$	$\{5_1, 7_1\}, \{0_2, 1_2\}$ (3 times)	$E_{14}$

New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
$(0_2 - 2_1, 4_1; 2_2 - 0_1, 3_1)$ $(0_2 - 3_1, 4_1; 3_2 - 0_1, 1_1)$	$(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(0_2 - 3_1, 1_2; 3_2 - 0_1, 1_1)$	$\{4_1, 0_2\}$ (twice), $\{0_2, 1_2\}$ (twice)	$E_{15}$
$(1_2 - 0_1, 9_1; 2_2 - 1_1, 2_1)$ $(9_1 - 7_1, 0_2; 3_2 - 4_1, 5_1)$ $(0_2 - 2_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 9_1, 1_2; 3_2 - 6_1, 8_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(9_1 - 0_2, 1_2; 3_2 - 4_1, 5_1)$ $(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$	$\{7_1, 9_1\}$ (twice), $\{0_2, 1_2\}$ (twice)	$E_{16}$
$(1_2 - 0_1, 5_1; 2_2 - 1_1, 2_1)$ $(0_2 - 2_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 5_1, 1_2; 3_2 - 6_1, 8_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$	$\{5_1, 7_1\}$ , $\{5_1, 1_2\}$ , $\{0_2, 1_2\}$ (twice)	$E_{17}$
$(1_2 - 0_1, 5_1; 2_2 - 1_1, 2_1)$ $(0_2 - 2_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 4_1, 1_2; 3_2 - 6_1, 8_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$	$\{4_1, 7_1\}$ , $\{5_1, 1_2\}$ , $\{0_2, 1_2\}$ (twice)	$E_{18}$
$(1_2 - 0_1, 9_1; 2_2 - 1_1, 2_1)$ $(9_1 - 8_1, 0_2; 3_2 - 4_1, 5_1)$ $(0_2 - 2_1, 7_1; 2_2 - 0_1, 3_1)$ $(7_1 - 5_1, 1_2; 3_2 - 6_1, 8_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(9_1 - 0_2, 1_2; 3_2 - 4_1, 5_1)$ $(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$	$\{5_1, 7_1\}$ , $\{8_1, 9_1\}$ , $\{0_2, 1_2\}$ (twice)	$E_{19}$
$(0_2 - 2_1, 7_1; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 7_1; 2_2 - 1_1, 2_1)$ $(7_1 - 4_1, 5_1; 3_2 - 6_1, 8_1)$	$(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(7_1 - 0_2, 1_2; 3_2 - 6_1, 8_1)$	$\{4_1, 7_1\}$ , $\{5_1, 7_1\}$ , $\{0_2, 1_2\}$ (twice)	$E_{20}$
$(1_2 - 0_1, 4_1; 2_2 - 1_1, 2_1)$ $(0_2 - 1_1, 2_1; 2_2 - 0_1, 3_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$	$\{1_1, 0_2\}$ , $\{4_1, 1_2\}$ , $\{0_2, 1_2\}$ (twice)	$E_{21}$
$(0_2 - 1_1, 2_1; 2_2 - 0_1, 3_1)$ $(0_2 - 2_1, 3_1; 3_2 - 0_1, 1_1)$	$(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$ $(0_2 - 3_1, 1_2; 3_2 - 0_1, 1_1)$	$\{1_1, 0_2\}$ , $\{2_1, 0_2\}$ , $\{0_2, 1_2\}$ (twice)	$E_{22}$
$(1_2 - 0_1, 4_1; 2_2 - 1_1, 2_1)$ $(0_2 - 2_1, 4_1; 2_2 - 0_1, 3_1)$	$(1_2 - 0_1, 0_2; 2_2 - 1_1, 2_1)$ $(0_2 - 2_1, 1_2; 2_2 - 0_1, 3_1)$	$\{4_1, 0_2\}$ , $\{4_1, 1_2\}$ , $\{0_2, 1_2\}$ (twice)	$E_{23}$

Table 4.10: Substitutions in the  $E$ -covering  $R \cup S$  to obtain different excess graphs for  $n = 14$

$S$  be the set consisting of the single path  $(2, 0, 1, 3, 4, 5)$ ,  $U$  be the set consisting of the single path  $(0, 1, 2, 3, 4, 5)$ , and  $V$  be the set consisting of the single path  $(2, 3, 0, 1, 4, 5)$ . Therefore,  $R \cup S$ ,  $R \cup U$ , and  $R \cup V$  form minimum  $P_6$ -coverings of  $K_{12}$  with the excess graphs  $E_5$  (with the edges  $\{0, 2\}$ ,  $\{1, 3\}$ ,  $\{3, 4\}$ , and  $\{4, 5\}$ ),  $E_6$  (with the edges  $\{1, 2\}$ ,  $\{2, 3\}$ ,  $\{3, 4\}$ , and  $\{4, 5\}$ ), and  $E_9$  (with the edges  $\{0, 3\}$ ,  $\{2, 3\}$ ,  $\{1, 4\}$ , and  $\{4, 5\}$ ), respectively.

In order to obtain the excess graph  $E_{12}$ , write  $K_{12} = K_8 \vee K_4$  and label the vertices of  $K_8 \vee K_4$  with a  $(\mathbb{Z}_8, \mathbb{Z}_4)$ -labeling. Let  $R$  be the set consisting of the following paths.

$$\begin{aligned} & (1_1, 0_1, 2_1, 6_1, 5_1, 4_1), (2_1, 1_1, 3_1, 5_1, 7_1, 0_1), (3_1, 2_1, 4_1, 7_1, 6_1, 1_1), \\ & (0_1, 3_1, 4_1, 1_1, 5_1, 2_1), (5_1, 0_1, 4_1, 6_1, 3_1, 7_1) \end{aligned}$$

In fact,  $R$  is a maximum  $P_6$ -packing of  $K_8$  with the edges  $\{0_1, 6_1\}$ ,  $\{1_1, 7_1\}$ , and  $\{2_1, 7_1\}$  as the leave graph. Now the set of vertices  $\{2_1, 3_1, 4_1, 5_1, 6_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , and the edges between these two sets, form a complete bipartite graph  $K_{5,4}$ , which has a  $P_6$ -decomposition,  $S$ , by Lemma 1.3.16. Also let  $U$  be the set consisting of the following paths.

$$\begin{aligned} & (1_1, 7_1, 2_2, 3_2, 0_1, 6_1), (0_1, 2_2, 1_2, 7_1, 1_1, 0_2), (0_1, 1_2, 3_2, 1_1, 7_1, 0_2), \\ & (0_1, 0_2, 3_2, 7_1, 1_1, 2_2), (2_1, 7_1, 1_1, 1_2, 0_2, 2_2) \end{aligned}$$

Therefore,  $R \cup S \cup U$  forms a minimum  $P_6$ -covering of  $K_{12}$  with the excess graph  $E_{12}$ . The edges of the excess graph are the edges  $\{1_1, 7_1\}$  used four times.

In order to achieve the remaining excess graphs we substitute some paths for others in the covering  $R \cup S \cup U$ . Table 4.11 illustrates these substitutions.

Case 14.  $T = P_6, n = 13$

The excess graph has two edges by Theorem 1.3.3. Hence, the possible excess graphs are  $K_2^2$ ,  $P_3$ , and  $2K_2$ . The leave graph in a maximum  $P_6$ -packing of  $K_{13}$  has three edges by Theorem 1.3.2. All simple graphs with three edges are achievable as the leave graph in a  $P_6$ -packing of  $K_{13}$  as we saw in Chapter 3. Label the vertices of  $K_{13}$  with the elements of  $\mathbb{Z}_{13}$  and let  $R$  be a maximum  $P_6$ -packing of  $K_{13}$  with the path  $(0, 1, 2, 3)$  as the leave graph. Also let  $S$  be the set consisting of the single path  $(0, 1, 2, 3, 4, 5)$  and  $U$  be the set consisting of the single path  $(4, 0, 1, 2, 3, 5)$ . Therefore,

New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
(2 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>1</sub> ) (0 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> ) (0 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>2</sub> )	{1 <sub>1</sub> , 7 <sub>1</sub> }, {2 <sub>1</sub> , 7 <sub>1</sub> }, {7 <sub>1</sub> , 0 <sub>2</sub> }, {7 <sub>1</sub> , 2 <sub>2</sub> }	$E_1$
(2 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>1</sub> ) (2 <sub>2</sub> , 1 <sub>1</sub> , 0 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 7 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> ) (0 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>2</sub> )	{0 <sub>1</sub> , 1 <sub>1</sub> }, {1 <sub>1</sub> , 7 <sub>1</sub> }, {2 <sub>1</sub> , 7 <sub>1</sub> }, {7 <sub>1</sub> , 0 <sub>2</sub> }	$E_2$
(2 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>1</sub> ) (1 <sub>1</sub> , 2 <sub>2</sub> , 0 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 7 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> ) (0 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>2</sub> )	{1 <sub>1</sub> , 7 <sub>1</sub> }, {2 <sub>1</sub> , 7 <sub>1</sub> }, {0 <sub>1</sub> , 2 <sub>2</sub> }, {7 <sub>1</sub> , 0 <sub>2</sub> }	$E_3$
(2 <sub>2</sub> , 1 <sub>1</sub> , 0 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 7 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>1</sub> ) (1 <sub>1</sub> , 3 <sub>2</sub> , 1 <sub>2</sub> , 0 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> )	(0 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>2</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> ) (0 <sub>1</sub> , 1 <sub>2</sub> , 3 <sub>2</sub> , 1 <sub>1</sub> , 7 <sub>1</sub> , 0 <sub>2</sub> )	{0 <sub>1</sub> , 1 <sub>1</sub> }, {1 <sub>1</sub> , 7 <sub>1</sub> }, {0 <sub>1</sub> , 0 <sub>2</sub> }, {7 <sub>1</sub> , 0 <sub>2</sub> }	$E_4$
(2 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (2 <sub>2</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>1</sub> , 2 <sub>2</sub> , 0 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 7 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (2 <sub>2</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>1</sub> ) (0 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>2</sub> )	{1 <sub>1</sub> , 2 <sub>1</sub> }, {1 <sub>1</sub> , 7 <sub>1</sub> }, {2 <sub>1</sub> , 7 <sub>1</sub> }, {0 <sub>1</sub> , 2 <sub>2</sub> }	$E_7$
(2 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (2 <sub>2</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 1 <sub>1</sub> , 2 <sub>1</sub> , 7 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (2 <sub>2</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> )	{1 <sub>1</sub> , 2 <sub>1</sub> }, {1 <sub>1</sub> , 7 <sub>1</sub> }, {2 <sub>1</sub> , 7 <sub>1</sub> }, {7 <sub>1</sub> , 0 <sub>2</sub> }	$E_8$
(0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>1</sub> ) (2 <sub>1</sub> , 6 <sub>1</sub> , 0 <sub>1</sub> , 3 <sub>2</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (1 <sub>1</sub> , 2 <sub>2</sub> , 0 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 7 <sub>1</sub> )	(0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> ) (1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 2 <sub>2</sub> )	{1 <sub>1</sub> , 7 <sub>1</sub> }, {2 <sub>1</sub> , 6 <sub>1</sub> }, {0 <sub>1</sub> , 2 <sub>2</sub> }, {7 <sub>1</sub> , 0 <sub>2</sub> }	$E_{10}$
(5 <sub>1</sub> , 6 <sub>1</sub> , 0 <sub>1</sub> , 3 <sub>2</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (1 <sub>1</sub> , 1 <sub>2</sub> , 0 <sub>2</sub> , 2 <sub>2</sub> , 2 <sub>1</sub> , 7 <sub>1</sub> ) (1 <sub>1</sub> , 3 <sub>2</sub> , 1 <sub>2</sub> , 0 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (2 <sub>1</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 1 <sub>2</sub> , 0 <sub>2</sub> , 2 <sub>2</sub> ) (0 <sub>1</sub> , 1 <sub>2</sub> , 3 <sub>2</sub> , 1 <sub>1</sub> , 7 <sub>1</sub> , 0 <sub>2</sub> )	{5 <sub>1</sub> , 6 <sub>1</sub> }, {1 <sub>1</sub> , 7 <sub>1</sub> }, {0 <sub>1</sub> , 0 <sub>2</sub> }, {2 <sub>1</sub> , 2 <sub>2</sub> }	$E_{11}$
(0 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> )	{1 <sub>1</sub> , 7 <sub>1</sub> } (3 times), {7 <sub>1</sub> , 0 <sub>2</sub> }	$E_{13}$
(7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> )	{1 <sub>1</sub> , 7 <sub>1</sub> } (3 times), {6 <sub>1</sub> , 0 <sub>2</sub> }	$E_{14}$
(0 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> )	{1 <sub>1</sub> , 7 <sub>1</sub> } (twice), {7 <sub>1</sub> , 0 <sub>2</sub> } (twice)	$E_{15}$
(1 <sub>1</sub> , 0 <sub>2</sub> , 0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> ) (1 <sub>1</sub> , 3 <sub>2</sub> , 1 <sub>2</sub> , 0 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> )	(0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> ) (0 <sub>1</sub> , 1 <sub>2</sub> , 3 <sub>2</sub> , 1 <sub>1</sub> , 7 <sub>1</sub> , 0 <sub>2</sub> )	{1 <sub>1</sub> , 7 <sub>1</sub> } (twice), {0 <sub>1</sub> , 0 <sub>2</sub> } (twice)	$E_{16}$
(0 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (1 <sub>1</sub> , 0 <sub>2</sub> , 0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> )	{1 <sub>1</sub> , 7 <sub>1</sub> } (twice), {0 <sub>1</sub> , 0 <sub>2</sub> }, {7 <sub>1</sub> , 0 <sub>2</sub> }	$E_{17}$
(1 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (1 <sub>1</sub> , 0 <sub>2</sub> , 0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> )	{1 <sub>1</sub> , 7 <sub>1</sub> } (twice), {0 <sub>1</sub> , 0 <sub>2</sub> }, {7 <sub>1</sub> , 1 <sub>2</sub> }	$E_{18}$
(7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> ) (1 <sub>1</sub> , 0 <sub>2</sub> , 0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> )	{1 <sub>1</sub> , 7 <sub>1</sub> } (twice), {0 <sub>1</sub> , 0 <sub>2</sub> }, {6 <sub>1</sub> , 2 <sub>2</sub> }	$E_{19}$
(7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> ) (1 <sub>1</sub> , 0 <sub>2</sub> , 0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> )	{1 <sub>1</sub> , 7 <sub>1</sub> } (twice), {0 <sub>1</sub> , 0 <sub>2</sub> }, {6 <sub>1</sub> , 0 <sub>2</sub> }	$E_{20}$
(0 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>2</sub> , 1 <sub>1</sub> , 0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> )	{0 <sub>1</sub> , 1 <sub>1</sub> }, {1 <sub>1</sub> , 7 <sub>1</sub> } (twice), {7 <sub>1</sub> , 0 <sub>2</sub> }	$E_{21}$
(1 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> )	{1 <sub>1</sub> , 7 <sub>1</sub> } (twice), {7 <sub>1</sub> , 0 <sub>2</sub> }, {7 <sub>1</sub> , 1 <sub>2</sub> }	$E_{22}$
(0 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 1 <sub>2</sub> , 3 <sub>2</sub> , 1 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> )	(1 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 3 <sub>2</sub> , 0 <sub>1</sub> , 6 <sub>1</sub> ) (0 <sub>1</sub> , 1 <sub>2</sub> , 3 <sub>2</sub> , 1 <sub>1</sub> , 7 <sub>1</sub> , 0 <sub>2</sub> )	{1 <sub>1</sub> , 7 <sub>1</sub> } (twice), {1 <sub>1</sub> , 0 <sub>2</sub> }, {7 <sub>1</sub> , 0 <sub>2</sub> }	$E_{23}$

Table 4.11: Substitutions in the  $P_6$ -covering  $RUSUU$  to obtain different excess graphs for  $n = 12$

$R \cup S$  and  $R \cup U$  are minimum  $P_6$ -coverings of  $K_{13}$  with the excess graphs  $P_3$  (with the edges  $\{3, 4\}$  and  $\{4, 5\}$ ) and  $2K_2$  (with the edges  $\{0, 4\}$  and  $\{3, 5\}$ ), respectively.

In order to obtain the excess graph  $K_2^2$ , write  $K_{13} = K_9 \vee K_4$  and label the vertices of  $K_9 \vee K_4$  with a  $(\mathbb{Z}_9, \mathbb{Z}_4)$ -labeling. By Theorem 1.3.2, the leave graph in a maximum  $P_6$ -packing of  $K_9$  has one edge. Let  $R$  be a maximum  $P_6$ -packing of  $K_9$  with the edge  $\{7_1, 8_1\}$  as the leave graph. The set of vertices  $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , and the edges between these two sets, form a complete bipartite graph  $K_{5,4}$ , which has a  $P_6$ -decomposition,  $S$ , by Lemma 1.3.16. Let  $U$  be the set consisting of the following paths.

$$\begin{aligned} &(1_2, 6_1, 2_2, 7_1, 8_1, 3_2), (5_1, 3_2, 1_2, 0_2, 7_1, 8_1), (5_1, 0_2, 2_2, 1_2, 7_1, 8_1), \\ &(8_1, 1_2, 5_1, 2_2, 3_2, 0_2), (7_1, 3_2, 6_1, 0_2, 8_1, 2_2) \end{aligned}$$

Therefore,  $R \cup S \cup U$  forms a minimum  $P_6$ -covering of  $K_{13}$  with the excess graph  $K_2^2$ . The edges of the excess graph are the edges  $\{7_1, 8_1\}$  used twice.

Case 15.  $T = P_6, n = 14$

The excess graph has four edges by Theorem 1.3.3. Hence, the possible excess graphs are those shown in Figure 4.1. Since a maximum  $P_6$ -packing of  $K_{14}$  has one edge by Theorem 1.3.2, the excess graphs  $E_5, E_6$ , and  $E_9$  can be achieved in a similar way as in Case 13.

In order to obtain the excess graph  $E_{12}$ , write  $K_{14} = K_{10} \vee K_4$  and label the vertices of  $K_{10} \vee K_4$  with a  $(\mathbb{Z}_{10}, \mathbb{Z}_4)$ -labeling. By Theorem 1.3.1,  $K_{10}$  has a  $P_6$ -decomposition,  $R$ . Moreover, the set of vertices  $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ , the set of vertices  $\{0_2, 1_2, 2_2, 3_2\}$ , and the edges between these two sets, form a complete bipartite graph  $K_{5,4}$ , which has a  $P_6$ -decomposition,  $S$ , by Lemma 1.3.16. Let  $U$  be the set consisting of the following paths.

$$\begin{aligned} &(5_1, 0_2, 1_2, 6_1, 2_2, 7_1), (8_1, 0_2, 1_2, 9_1, 3_2, 7_1), (5_1, 3_2, 6_1, 0_2, 1_2, 2_2), \\ &(5_1, 2_2, 0_2, 1_2, 8_1, 3_2), (8_1, 2_2, 9_1, 0_2, 1_2, 3_2), (5_1, 1_2, 7_1, 0_2, 3_2, 2_2) \end{aligned}$$

Therefore,  $R \cup S \cup U$  forms a minimum  $P_6$ -covering of  $K_{14}$  with the excess graph  $E_{12}$ . The edges of the excess graph are the edges  $\{0_2, 1_2\}$  used four times.

In order to achieve the remaining excess graphs, we substitute some paths for

others in the excess graph  $R \cup S \cup U$ . These substitutions are illustrated in Table 4.12.

■



New tree(s)	Previous tree(s)	Edges of the excess graph	Excess
(5 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> ) (5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> ) (8 <sub>1</sub> , 2 <sub>2</sub> , 9 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 2 <sub>2</sub> ) (8 <sub>1</sub> , 2 <sub>2</sub> , 9 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 3 <sub>2</sub> )	{7 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 3 <sub>2</sub> }	$E_1$
(5 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> ) (8 <sub>1</sub> , 2 <sub>2</sub> , 9 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 1 <sub>2</sub> ) (0 <sub>2</sub> , 6 <sub>1</sub> , 3 <sub>2</sub> , 5 <sub>1</sub> , 1 <sub>2</sub> , 2 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (8 <sub>1</sub> , 2 <sub>2</sub> , 9 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 3 <sub>2</sub> ) (5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 2 <sub>2</sub> )	{5 <sub>1</sub> , 1 <sub>2</sub> }, {7 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 3 <sub>2</sub> }	$E_2$
(5 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> ) (8 <sub>1</sub> , 2 <sub>2</sub> , 9 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 1 <sub>2</sub> ) (0 <sub>2</sub> , 6 <sub>1</sub> , 3 <sub>2</sub> , 5 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (8 <sub>1</sub> , 2 <sub>2</sub> , 9 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 3 <sub>2</sub> ) (5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 2 <sub>2</sub> )	{5 <sub>1</sub> , 2 <sub>2</sub> }, {7 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 3 <sub>2</sub> }	$E_3$
(0 <sub>2</sub> , 5 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> ) (8 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> , 3 <sub>2</sub> , 9 <sub>1</sub> , 1 <sub>2</sub> ) (0 <sub>2</sub> , 6 <sub>1</sub> , 3 <sub>2</sub> , 5 <sub>1</sub> , 1 <sub>2</sub> , 2 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (7 <sub>1</sub> , 3 <sub>2</sub> , 9 <sub>1</sub> , 1 <sub>2</sub> , 0 <sub>2</sub> , 8 <sub>1</sub> ) (5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 2 <sub>2</sub> )	{5 <sub>1</sub> , 7 <sub>1</sub> }, {5 <sub>1</sub> , 1 <sub>2</sub> }, {7 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_4$
(2 <sub>2</sub> , 1 <sub>2</sub> , 5 <sub>1</sub> , 0 <sub>2</sub> , 6 <sub>1</sub> , 3 <sub>2</sub> ) (0 <sub>2</sub> , 2 <sub>2</sub> , 5 <sub>1</sub> , 3 <sub>2</sub> , 8 <sub>1</sub> , 1 <sub>2</sub> ) (0 <sub>2</sub> , 9 <sub>1</sub> , 2 <sub>2</sub> , 8 <sub>1</sub> , 3 <sub>2</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 2 <sub>2</sub> ) (5 <sub>1</sub> , 2 <sub>2</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 8 <sub>1</sub> , 3 <sub>2</sub> ) (8 <sub>1</sub> , 2 <sub>2</sub> , 9 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 3 <sub>2</sub> )	{5 <sub>1</sub> , 0 <sub>2</sub> }, {5 <sub>1</sub> , 1 <sub>2</sub> }, {8 <sub>1</sub> , 3 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_7$
(2 <sub>2</sub> , 1 <sub>2</sub> , 5 <sub>1</sub> , 0 <sub>2</sub> , 6 <sub>1</sub> , 3 <sub>2</sub> ) (0 <sub>2</sub> , 2 <sub>2</sub> , 5 <sub>1</sub> , 3 <sub>2</sub> , 8 <sub>1</sub> , 1 <sub>2</sub> ) (8 <sub>1</sub> , 2 <sub>2</sub> , 9 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 2 <sub>2</sub> ) (5 <sub>1</sub> , 2 <sub>2</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 8 <sub>1</sub> , 3 <sub>2</sub> ) (8 <sub>1</sub> , 2 <sub>2</sub> , 9 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 3 <sub>2</sub> )	{5 <sub>1</sub> , 0 <sub>2</sub> }, {5 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 3 <sub>2</sub> }	$E_8$
(0 <sub>2</sub> , 5 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> ) (5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> ) (0 <sub>2</sub> , 9 <sub>1</sub> , 2 <sub>2</sub> , 8 <sub>1</sub> , 3 <sub>2</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 2 <sub>2</sub> ) (8 <sub>1</sub> , 2 <sub>2</sub> , 9 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 3 <sub>2</sub> )	{5 <sub>1</sub> , 7 <sub>1</sub> }, {8 <sub>1</sub> , 3 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 2 <sub>2</sub> }	$E_{10}$
(0 <sub>2</sub> , 5 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> ) (0 <sub>2</sub> , 9 <sub>1</sub> , 2 <sub>2</sub> , 8 <sub>1</sub> , 3 <sub>2</sub> , 1 <sub>2</sub> ) (0 <sub>2</sub> , 6 <sub>1</sub> , 3 <sub>2</sub> , 5 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> ) (6 <sub>1</sub> , 2 <sub>2</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 8 <sub>1</sub> , 3 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (8 <sub>1</sub> , 2 <sub>2</sub> , 9 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 3 <sub>2</sub> ) (5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 2 <sub>2</sub> ) (5 <sub>1</sub> , 2 <sub>2</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 8 <sub>1</sub> , 3 <sub>2</sub> )	{5 <sub>1</sub> , 7 <sub>1</sub> }, {6 <sub>1</sub> , 2 <sub>2</sub> }, {8 <sub>1</sub> , 3 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> }	$E_{11}$
(5 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> )	{7 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (3 times)	$E_{13}$
(0 <sub>2</sub> , 5 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> )	{5 <sub>1</sub> , 7 <sub>1</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (3 times)	$E_{14}$
(5 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> ) (8 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> , 3 <sub>2</sub> , 9 <sub>1</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (7 <sub>1</sub> , 3 <sub>2</sub> , 9 <sub>1</sub> , 1 <sub>2</sub> , 0 <sub>2</sub> , 8 <sub>1</sub> )	{7 <sub>1</sub> , 0 <sub>2</sub> } (twice), {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{15}$
(0 <sub>2</sub> , 2 <sub>2</sub> , 5 <sub>1</sub> , 3 <sub>2</sub> , 8 <sub>1</sub> , 1 <sub>2</sub> ) (0 <sub>2</sub> , 7 <sub>1</sub> , 1 <sub>2</sub> , 5 <sub>1</sub> , 3 <sub>2</sub> , 2 <sub>2</sub> ) (8 <sub>1</sub> , 2 <sub>2</sub> , 9 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 2 <sub>2</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 8 <sub>1</sub> , 3 <sub>2</sub> ) (5 <sub>1</sub> , 1 <sub>2</sub> , 7 <sub>1</sub> , 0 <sub>2</sub> , 3 <sub>2</sub> , 2 <sub>2</sub> ) (8 <sub>1</sub> , 2 <sub>2</sub> , 9 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 3 <sub>2</sub> )	{5 <sub>1</sub> , 3 <sub>2</sub> } (twice), {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{16}$
(0 <sub>2</sub> , 5 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> ) (8 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> , 3 <sub>2</sub> , 9 <sub>1</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (7 <sub>1</sub> , 3 <sub>2</sub> , 9 <sub>1</sub> , 1 <sub>2</sub> , 0 <sub>2</sub> , 8 <sub>1</sub> )	{5 <sub>1</sub> , 7 <sub>1</sub> }, {7 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{17}$
(5 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> ) (0 <sub>2</sub> , 6 <sub>1</sub> , 3 <sub>2</sub> , 5 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 2 <sub>2</sub> )	{5 <sub>1</sub> , 2 <sub>2</sub> }, {7 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{18}$
(0 <sub>2</sub> , 5 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> ) (0 <sub>2</sub> , 9 <sub>1</sub> , 2 <sub>2</sub> , 8 <sub>1</sub> , 3 <sub>2</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (8 <sub>1</sub> , 2 <sub>2</sub> , 9 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 3 <sub>2</sub> )	{5 <sub>1</sub> , 7 <sub>1</sub> }, {8 <sub>1</sub> , 3 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{19}$
(0 <sub>2</sub> , 5 <sub>1</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> ) (0 <sub>2</sub> , 6 <sub>1</sub> , 3 <sub>2</sub> , 5 <sub>1</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 2 <sub>2</sub> )	{5 <sub>1</sub> , 7 <sub>1</sub> }, {5 <sub>1</sub> , 2 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{20}$
(5 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> ) (7 <sub>1</sub> , 3 <sub>2</sub> , 9 <sub>1</sub> , 1 <sub>2</sub> , 8 <sub>1</sub> , 0 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (7 <sub>1</sub> , 3 <sub>2</sub> , 9 <sub>1</sub> , 1 <sub>2</sub> , 0 <sub>2</sub> , 8 <sub>1</sub> )	{7 <sub>1</sub> , 0 <sub>2</sub> }, {8 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{21}$
(5 <sub>1</sub> , 0 <sub>2</sub> , 7 <sub>1</sub> , 2 <sub>2</sub> , 6 <sub>1</sub> , 1 <sub>2</sub> ) (5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 2 <sub>2</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 6 <sub>1</sub> , 2 <sub>2</sub> , 7 <sub>1</sub> ) (5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 2 <sub>2</sub> )	{7 <sub>1</sub> , 0 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice), {0 <sub>2</sub> , 2 <sub>2</sub> }	$E_{22}$
(2 <sub>2</sub> , 1 <sub>2</sub> , 5 <sub>1</sub> , 0 <sub>2</sub> , 6 <sub>1</sub> , 3 <sub>2</sub> ) (0 <sub>2</sub> , 2 <sub>2</sub> , 5 <sub>1</sub> , 3 <sub>2</sub> , 8 <sub>1</sub> , 1 <sub>2</sub> )	(5 <sub>1</sub> , 3 <sub>2</sub> , 6 <sub>1</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 2 <sub>2</sub> ) (5 <sub>1</sub> , 2 <sub>2</sub> , 0 <sub>2</sub> , 1 <sub>2</sub> , 8 <sub>1</sub> , 3 <sub>2</sub> )	{5 <sub>1</sub> , 0 <sub>2</sub> }, {5 <sub>1</sub> , 1 <sub>2</sub> }, {0 <sub>2</sub> , 1 <sub>2</sub> } (twice)	$E_{23}$

Table 4.12: Substitutions in the  $P_6$ -covering  $RUSUU$  to obtain different excess graphs for  $n = 14$

# Chapter 5

## Conclusions and Future Work

In this thesis, we constructed the spectrum of leave and excess graphs for trees that have five edges or less. In the future, we will consider the spectrum of leave graphs for trees that have a higher number of edges.

A cyclic  $G$ -decomposition of the complete graph  $K_n$  is a  $G$ -decomposition of  $K_n$  whose automorphism group contains a cycle of length  $n$ . Figure 5.1 illustrates a cyclic  $K_3$ -decomposition of  $K_7$ .

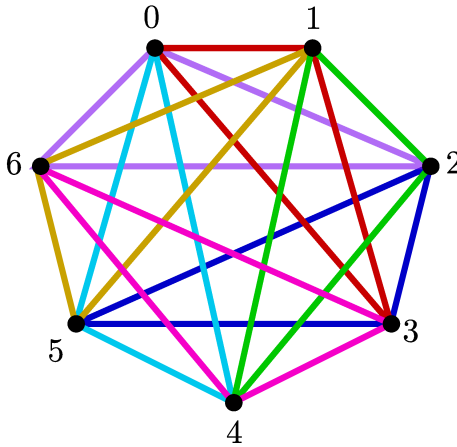


Figure 5.1: A cyclic  $K_3$ -decomposition of  $K_7$

It is natural to define cyclic packings and coverings in a similar way. Cyclic decompositions of the complete graph have been studied for many graphs (see [7], [18], [31], [41], and [42]). In the future, we might consider to improve our proofs as

well as to establish new results by finding cyclic tree-packings and tree-coverings of the complete graph.

As stated in the first chapter, Roditty proved that except for small integers  $n$ , the  $T$ -packing and  $T$ -covering numbers of  $K_n$  are  $\left\lfloor \frac{n(n-1)}{2i} \right\rfloor$  and  $\left\lceil \frac{n(n-1)}{2i} \right\rceil$ , respectively, where  $T$  is any tree with  $i \leq 6$  edges. It is of interest to find the  $T$ -packing and  $T$ -covering numbers of the complete graph for trees  $T$  with more edges. However, solving the spectrum problem for packing and covering (especially the covering) for trees with more than five edges will be difficult, since the number of edges in the leave and excess graphs are larger. For example, consider  $S_6$  as a tree with six edges. For  $n = 14$ , the excess graph in any minimum  $S_6$ -covering of  $K_n$  has five edges and there are 48 possible excess graphs (see Figure 5.2).

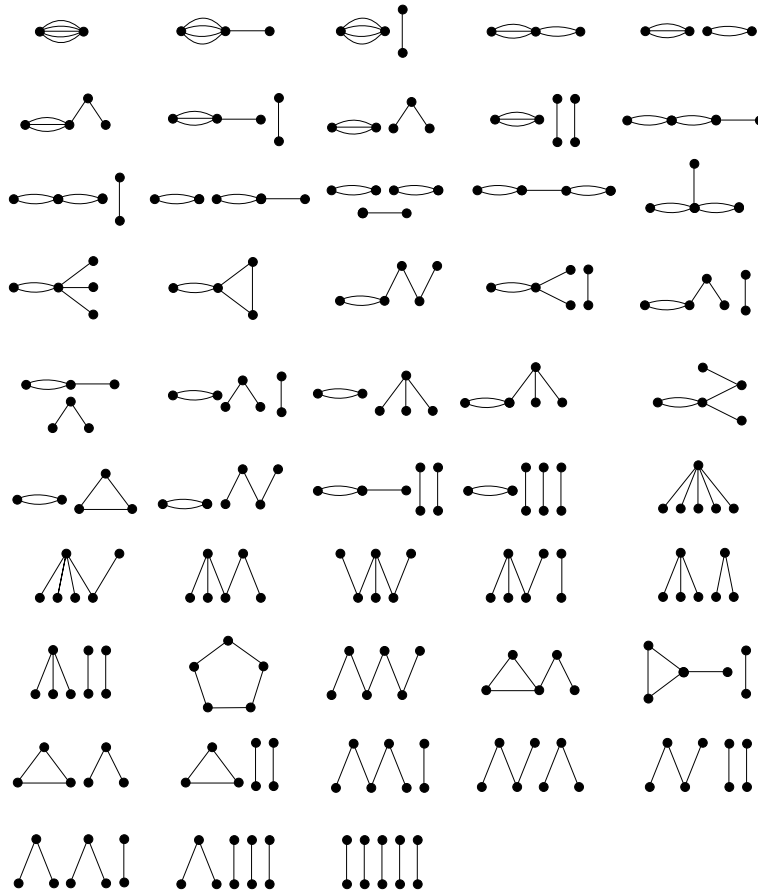


Figure 5.2: All possible excess graphs with five edges

Another direction that could be pursued is to consider decomposition (packing or

covering) of the complete graph with different types of trees. For instance, consider the complete graph  $K_{16}$  and write  $K_{16} = K_{10} \vee K_6$ . Since the graphs  $K_{10}$ ,  $K_6$ , and  $K_{10,6}$  have a  $D$ -decomposition,  $S_3$ -decomposition, and  $E$ -decomposition respectively, The graph  $K_{16}$  can be decomposed with the trees  $S_3$ ,  $D$ , and  $E$ . This idea might lead to a proof of the conjecture made in 1978 by Gyrfas and Lehel [21]. They conjectured that the complete graph  $K_n$  can be decomposed into any collection of trees  $T_1, T_2, \dots, T_{n-1}$ , where each  $T_i$  is a tree with  $i$  edges.

In 1975, Yamamoto proved that the necessary and sufficient conditions for the existence of an  $S_k$ -decomposition of  $K_n$  are  $n = 1$  or  $n \geq 2k$ , and  $n(n-1) \equiv 0 \pmod{2k}$  [51]. In 2014, Hoffman solved the packing and covering problems for any  $k$ -star [22]. In fact, he proved that for  $n \geq 2k$ , the number of  $k$ -stars in a maximum  $S_k$ -packing of  $K_n$  is  $\left\lfloor \frac{n(n-1)}{2k} \right\rfloor$ , and a star is always achievable as the leave graph. We might consider generalizing this result for other possible leave graphs.

A *maximal  $G$ -packing* of  $H$  is a  $G$ -packing of  $H$  in which the leave graph contains no subgraph  $G$ . The difference between the maximal and maximum packing is that in a maximum packing the leave graph has the smallest possible number of edges, while in a maximal packing the leave graph can have any number of edges as long as it does not contain any subgraphs  $G$ . For example, Figure 5.3 demonstrates a maximal  $K_3$ -packing of  $K_6$  which is not maximum. Another subject to consider is the maximal packing of the complete graph with small trees.

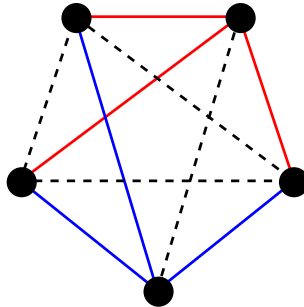


Figure 5.3: A maximal  $S_3$ -packing of  $K_5$

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